

CONSTRAINED PERCOLATION ON \mathbb{Z}^2

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ABSTRACT. We study a constrained percolation process on \mathbb{Z}^2 , and prove the almost sure nonexistence of infinite clusters and contours for a large class of probability measures. Unlike the unconstrained case, in the constrained case no stochastic monotonicity is known. We prove the nonexistence of infinite clusters and contours for the constrained percolation by developing new combinatorial techniques which make use of the planar duality and symmetry.

Applications include the almost sure nonexistence of infinite homogeneous clusters for the critical dimer model on the square-octagon lattice, as well as the almost sure nonexistence of infinite monochromatic contours and infinite clusters for the critical XOR Ising model on the square grid. By relaxing the symmetric condition of the underlying Gibbs measure on the constrained percolation process, we prove that there exists at most one infinite monochromatic contour for the non-critical XOR Ising model.

1. INTRODUCTION

1.1. Background. A fundamental question in the theory of percolation and interacting particle system is whether there exists an infinite connected set in which each item has the same state (infinite cluster). As a defining property of phase transition, it is, however, a challenging problem to determine whether an infinite cluster exists when the phase transition occurs (at criticality). Regarding this problem, there are numerous, spectacular works, see, for example, [20, 24, 19, 4, 12], for an incomplete list in this direction.

Instead of studying the well-known i.i.d. Bernoulli percolation (see [13]), in this paper, we study a constrained percolation. Compared to the unconstrained case, the constrained percolation studies probability measures restricted on a “subspace” of the unconstrained sample space, i.e., only configurations satisfying certain local constraints have positive probability to occur. The constrained percolation is very common in nature and has been an interesting problem for mathematicians and physicists for long. For example, the perfect matching (dimer model), i.e. subset of edges such that each vertex has exactly one incident present edge (see [21]); the 1-2 model, i.e. subset of edges on a cubic graph such that each vertex has one or two incident present edges (see [17]); the 6-vertex model, i.e., configurations of edge orientations on a degree-4 graph such that each vertex has exactly two incoming edges, and two outgoing edges (see [3]); and some general vertex models which can be transformed to dimer models on decorated graphs via the holographic algorithm ([34, 8, 32, 25]).

Phase transitions of certain constrained percolation problems have been studied extensively, see, for example, [23] for the dimer model, and [16] for the 1-2 model. The integrability properties of these constrained percolation problems make it possible to exactly compute the finite-dimensional distribution, and to study the corresponding analyticity

property. The critical parameter, i.e. parameter where discontinuity of the correlation is observed, can usually be solved from an explicit algebraic equation.

Although there are many nice results to describe the phase transition of constrained percolation problem by a microscopic observable, e.g. spin-spin correlation, up to now, very few papers study the phase transition of constrained percolation from a macroscopic perspective, e.g. the existence of an infinite cluster. Even though sometimes we know the phase transition exists with respect to a macroscopic observable, the exact value of the critical parameter is unknown ([27]), and it is not known if the critical parameter in the macroscopic sense coincides with the critical parameter in the microscopic sense, except for very few special models, such as the 2-dimensional Ising model ([2, 28, 26]). One difficulty for the constrained percolation problem is that there is usually no stochastic monotonicity, see also [33, 30, 18].

1.2. Constrained Percolation. In this paper, we study a constrained site percolation problem on the 2-dimensional lattice \mathbb{Z}^2 . Let G be the 2-dimensional square grid, i.e. $G = (\mathbb{Z}^2, E)$. The vertex set of G consists of all points (m, n) on the plane, such that $m, n \in \mathbb{Z}$. Two vertices (m, n) and (m', n') are connected by an edge in E if and only if $|m - m'| + |n - n'| = 1$.

Obviously each face of G is a unit square. We say two faces of G are incident if they share an edge. Let f be a face of G , and (m_f, n_f) be the coordinate of the vertex at the lower right corner of f . We color f by white if $m_f + n_f$ is even. If $m_f + n_f$ is odd, we color f by black. This way each square of G is colored by either black or white, such that black squares are incident only to white squares and vice versa.

We consider the site percolation on G , i.e., the state space is $\{0, 1\}^{\mathbb{Z}^2}$. We impose the following constraint on site configurations

- Around each black square, there are 6 different configurations $\{0000\}$, $\{1111\}$, $\{0011\}$, $\{1100\}$, $\{0110\}$, $\{1001\}$, where the digits from the left to the right correspond to vertices winding counterclockwise around the black square, starting from the lower right corner.

Note that in the unconstrained case, around each black square, there are 16 different configurations; only 6 of which are allowed in the constrained case. See Figure 1 for local configurations of the constrained percolation around a black square.

Let $\Omega' \subseteq \{0, 1\}^{\mathbb{Z}^2}$ be the set of all configurations satisfying the constraint above.

Let μ be a probability measure on Ω' . Throughout this paper, we assume μ satisfies the following conditions

Assumption 1.1.

- μ is $2\mathbb{Z} \times 2\mathbb{Z}$ translation invariant.

- μ is $2\mathbb{Z} \times 2\mathbb{Z}$ ergodic; i.e., any $2\mathbb{Z} \times 2\mathbb{Z}$ translation invariant event has μ probability 0 or 1.

Moreover, we may assume μ satisfies further conditions below

Assumption 1.2.

- μ is \mathcal{H} translation invariant, where \mathcal{H} is the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(1, 1)$ and $(1, -1)$.

- μ is symmetric, i.e. let v_1, \dots, v_k be k distinct vertices, and n, n^* be two k -digit binary numbers satisfying $n + n^* = 2^k - 1$; in other words, n^* is the binary number

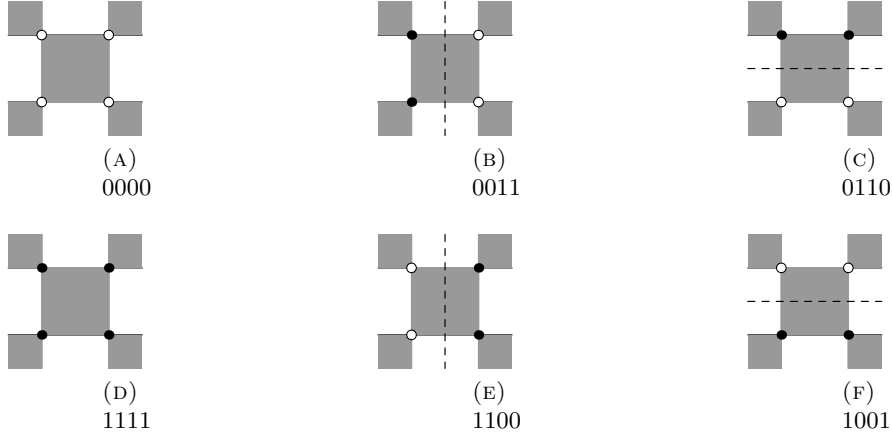


FIGURE 1. local configurations of the constrained percolation around a black square

obtained from n by replacing every 1 by 0, and every 0 by 1, then

$$\mu(\omega \in \Omega' : \omega|_{v_1, \dots, v_k} = n) = \mu(\omega \in \Omega' : \omega|_{v_1, \dots, v_k} = n^*),$$

where $\omega|_{v_1, \dots, v_k}$ is the restriction of the configuration ω on vertices v_1, \dots, v_k , which can be represented by a k -digit binary number, and the i th digit ($1 \leq i \leq k$) denotes the state of v_i (“0” or “1”).

1.3. Contours and Clusters. We will construct two auxiliary square grids, \mathbb{L} and \mathbb{L}^* , whose vertices are white faces of the original square grid G . The primal (resp. dual) auxiliary square grid \mathbb{L} (resp. \mathbb{L}^*) has vertices located at points $(m - \frac{1}{2}, n + \frac{1}{2})$ of the plane, in which both m and n are even (resp. odd). Two vertices $(m - \frac{1}{2}, n + \frac{1}{2})$, $(m' - \frac{1}{2}, n' + \frac{1}{2})$ of \mathbb{L} (resp. \mathbb{L}^*) are connected by an edge of \mathbb{L} (resp. \mathbb{L}^*) if and only if $|m - m'| + |n - n'| = 2$. Evidently each face of \mathbb{L} or \mathbb{L}^* is a square of side length 2.

Moreover, each black face of G corresponds to an edge of \mathbb{L} (in the sense that the black face of G and the corresponding edge of \mathbb{L} share a center), also an edge of \mathbb{L}^* . The two edges of \mathbb{L} and \mathbb{L}^* , corresponding to the same black face of G , are perpendicular to each other. Each site configuration in Ω' corresponds to a configuration in $\{0, 1\}^{E(\mathbb{L}) \cup E(\mathbb{L}^*)}$, where $E(\mathbb{L})$ (resp. $E(\mathbb{L}^*)$) is the edge set of \mathbb{L} (resp. \mathbb{L}^*). Locally, $\{0000\}$ and $\{1111\}$ correspond to the configuration that neither the primal edge nor the dual edge corresponding to the black face are present; $\{1001\}$ and $\{0110\}$ correspond to the configuration that only the horizontal edge crossing the black square of G is present; $\{0011\}$ and $\{1100\}$ correspond to the configuration that only the vertical edge is present, see Figure 1, where the present edges of \mathbb{L} or \mathbb{L}^* are represented by dashed lines. Since the present edges of \mathbb{L} or \mathbb{L}^* are boundaries separating the state “0” and the state “1”, we conclude that each vertex of \mathbb{L} or \mathbb{L}^* has an even number of incident present edges. Moreover, the present edges of \mathbb{L} and \mathbb{L}^* can never intersect.

We define contour configurations to be configurations in $\{0, 1\}^{E(\mathbb{L}) \cup E(\mathbb{L}^*)}$, where an edge $e \in E(\mathbb{L}) \cup E(\mathbb{L}^*)$ is present if and only if it has state “1”, satisfying the following conditions

I each vertex in \mathbb{L} or \mathbb{L}^* has an even number of incident present edges;

II present edges in $E(\mathbb{L})$ and present edges in $E(\mathbb{L}^*)$ do not intersect.

Let $\Phi' \subset \{0, 1\}^{E(\mathbb{L}) \cup E(\mathbb{L}^*)}$ be the set of all contour configurations satisfying the conditions above. According to previous discussions, there is a 2-to-1 correspondence between constrained site configurations in Ω' and contour configurations in Φ' . If $\omega \in \Omega'$, let ω^* be the configuration obtained from ω by changing the state of every vertex, i.e., if a vertex has state “1” in ω , change it to “0” in ω^* , and vice versa, then ω and ω^* correspond to the same contour configuration in Φ' .

Let $\phi \in \Phi'$ be a contour configuration. Each connected component of present edges in ϕ is called a contour of ϕ . Since present primal edges and dual primal edges do not intersect in a contour configuration, either all the edges in a contour are primal edges (edges of \mathbb{L}), or all the edges in a contour are dual edges (edges of \mathbb{L}^*). A contour is a primal (resp. dual) contour if it consists of edges of \mathbb{L} (resp. edges of \mathbb{L}^*). A (primal or dual) contour is called finite (resp. infinite) if it consists of finitely many (resp. infinitely many) edges (of \mathbb{L} or \mathbb{L}^*).

Let $V \subseteq \mathbb{Z}^2$ be a set of vertices in \mathbb{Z}^2 . We say V is connected if for any $v_1, v_2 \in V$, there exists a sequence $u_0(=v_1), u_1, \dots, u_{n-1}, u_n(=v_2)$, such that $v_i \in V$, for $0 \leq i \leq n$, and v_i and v_{i-1} are incident in G for $1 \leq i \leq n$.

Let $\omega \in \Omega'$. A cluster of ω is a largest connected set of vertices of G , in which every vertex has the same state. If all the vertices in a cluster have the state “0” (resp. “1”), we call the cluster a “0”-cluster (resp. “1”-cluster). We say a cluster is finite (resp. infinite) if the total number of vertices in the cluster is finite (resp. infinite). Each cluster of the site configuration on G is also a connected component of \mathbb{Z}^2 divided by the corresponding contour configuration on $E(\mathbb{L}) \cup E(\mathbb{L}^*)$.

Let C be a contour, and D be a cluster. We say a contour C is adjacent to a cluster D if there exists $e \in C$, where e is an edge in $E(\mathbb{L}) \cup E(\mathbb{L}^*)$, and a vertex $v \in D$, where v is a vertex in \mathbb{Z}^2 , such that $\text{dist}(v, e) = \frac{1}{2}$.

Let $G_* = (\mathbb{Z}^2, E_*)$ be a graph with vertex set \mathbb{Z}^2 . Two vertices $(m, n), (m', n') \in \mathbb{Z}^2$ are connected by an edge in E_* if and only if

$$(m' - m, n' - n) \in \{(\pm 1, 0), (0, \pm 1), (1, \pm 1), (-1, \pm 1)\}$$

Note that, for example, the two vertices $(0, 0)$ and $(1, 1)$ are not adjacent in G , but adjacent in G_* .

A $*$ -cluster of a configuration $\omega \in \Omega'$ is a largest connected set of vertices in G_* , in which every vertex has the same state. Each cluster of ω is a subset of a $*$ -cluster of ω . Similarly we define the “0”- $*$ -cluster, “1”- $*$ -cluster, finite $*$ -cluster, and infinite $*$ -cluster. In particular, if there is an infinite cluster in ω , then there is an infinite $*$ -cluster in ω , but not vice versa.

Let \mathbb{R}^2 be the plane. Throughout this paper, we say a measurable subset U of \mathbb{R}^2 is finite (resp. infinite) if $U \cap \mathbb{Z}^2$ has finitely (resp. infinitely) many vertices.

Let μ be a Gibbs measure on Ω' , satisfying Assumption 1.1. Note that μ induces a probability measure ν on contour configurations in Φ' . More precisely, the correspondence can be described as follows. Let $e_1, \dots, e_n \in E(\mathbb{L}) \cup E(\mathbb{L}^*)$. Let S_{e_1, \dots, e_n} be the event that all the edges e_1, \dots, e_n are present in a contour configuration in Φ' . According to the construction of \mathbb{L} and \mathbb{L}^* in Sect. 1.3, we know that for $1 \leq i \leq n$, e_i shares the center

with a unique black square Q_i of G . Assume Q_i has four vertices $v_i^1, v_i^2, v_i^3, v_i^4$, such that v_i^1 and v_i^2 are on one side of e_i , while v_i^3 and v_i^4 are on the other side of v_i . Let T_{e_1, \dots, e_n} be the event satisfying all the following conditions

- I for $1 \leq i \leq n$, v_i^1 and v_i^2 have the same state;
- II for $1 \leq i \leq n$, v_i^3 and v_i^4 have the same state;
- III for $1 \leq i \leq n$, v_i^1 and v_i^3 have different states;

Then we have

$$(1) \quad \nu(S_{e_1, \dots, e_n}) = \mu(T_{e_1, \dots, e_n}).$$

Note that (1) determines the finite dimensional distributions; and ν is the unique Kolmogorov extension obtained from the finite dimensional distribution given by (1).

Let ν_1 (resp. ν_2) be the corresponding marginal distribution of ν on bond configurations of \mathbb{L} (resp. \mathbb{L}^*). Let Φ'_1 (resp. Φ'_2) be the state space consisting of bond configurations of \mathbb{L} (resp. \mathbb{L}^*) satisfying the condition that each vertex has an even number of incident present edges, and let \mathcal{F}_p (resp. \mathcal{F}_d) be the σ -algebra on Φ'_1 (resp. Φ'_2) generated by subsets of Φ'_1 (resp. Φ'_2) depending on states of finitely many edges. Then we have probability spaces $(\Phi'_1, \mathcal{F}_p, \nu_1)$ and $(\Phi'_2, \mathcal{F}_d, \nu_2)$.

Assumption 1.3. Assume ν_1 has finite energy in the following sense: let $F = \cup_{i=1}^k S_i$ be the union of finitely many primal squares S_1, \dots, S_k in \mathbb{L} . Let ∂F be the boundary of F , consisting of edges of \mathbb{L} . Let $Q \in \Phi'_1$. Define Q_F as follows: if $e \notin \partial F$, then $Q_F(e)$ and $Q(e)$ have the same state, i.e. either both are present or both are absent; if $e \in \partial F$, then $Q_F(e)$ is present if and only if $Q(e)$ is absent. Let $E \in \mathcal{F}_p$ be an event, define

$$(2) \quad E_F = \{Q_F : Q \in E\}.$$

Then for any F consisting of finitely many primal squares, $\nu_1(E_F) > 0$, whenever $\nu_1(E) > 0$.

Note that, for each $Q \in \Phi'_1$, the corresponding Q_F defined in Assumption 1.3 is still in Φ'_1 , for any F consisting of finitely many primal squares; see also Lemma 3.1. Moreover, if $E \in \mathcal{F}_p$, we have $E_F \in \mathcal{F}_p$, where E_F is defined in (2).

1.4. Main Results. The main results of this paper are summarized in the following theorems

Theorem 1.4. Let μ be a probability measure on the constrained percolation state space Ω' , satisfying Assumptions 1.1, 1.2, let \mathcal{A} be the event that there exists a unique infinite cluster consisting of “1”’s, then

$$\mu(\mathcal{A}) = 0.$$

Theorem 1.5. Let μ be a probability measure on Ω' , satisfying Assumptions 1.1, 1.2, 1.3. Then μ -a.s.

- I there are neither infinite “0”-clusters nor infinite “1”-clusters;
- II there are neither infinite primal contours nor infinite dual contours;
- III there are no infinite *-clusters.

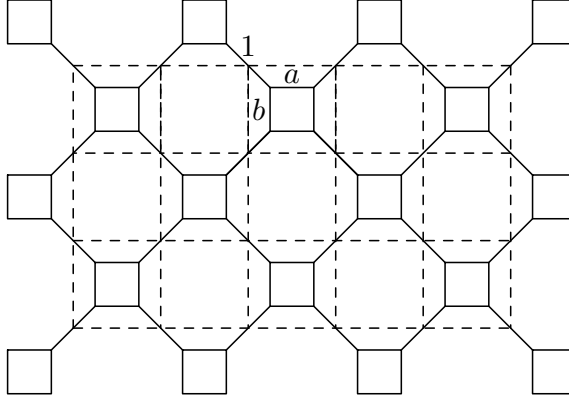


FIGURE 2. Square-Octagon Lattice

Theorem 1.6. *Let μ be a probability measure on Ω' satisfying the Assumptions 1.1, 1.2, 1.3. Let ν_1 (resp. ν_2) be the corresponding marginal distributions on bond configurations in Φ'_1 (resp. Φ'_2). Let ϕ_p (resp. ϕ_d) be the union of all primal (resp. dual) contours. The ν_1 -a.s. (resp. ν_2 -a.s.) $\mathbb{R}^2 \setminus \phi_p$ (resp. $\mathbb{R}^2 \setminus \phi_d$) has no infinite components.*

Theorem 1.7. *Let μ be a probability measure on the constrained percolation state space Ω' , satisfying Assumptions 1.1, 1.3, then μ -a.s. there is at most one infinite primal contour.*

Here is the organization of this paper. In Sect. 2, we study examples of the constrained percolation measure μ satisfying Assumptions 1.1, 1.2, 1.3, and applications of Theorems 1.4-1.7 to these examples. In Sect. 3, we prove combinatorial results regarding the configurations of contours and clusters. In Sect. 4, we prove Theorem 1.4. In Sect. 5, we prove Theorems 1.5 and 1.6. In Sect. 6, we prove Theorem 1.7.

2. EXAMPLES OF CONSTRAINED PERCOLATION MEASURE

In this section, we apply Theorems 1.4-1.7 to perfect matchings on the square-octagon lattice, as well as the XOR Ising model on the square grid, to obtain results about infinite clusters and infinite contours in these well-known, specific models.

Consider perfect matchings on the square-octagon lattice; namely, each perfect matching, or dimer configuration on the square-octagon lattice is a subset of edges such that each vertex of the square-octagon lattice is incident to exactly one edge in the subset. There are two types of edges for the square-octagon lattice: Type-I edges are edges of the squares and Type-II edges are edges of the octagons but not edges of the squares. See Figure 2 for a picture of the square-octagon lattice.

Recall that $G = (\mathbb{Z}^2, E)$ is the square grid, whose faces are unit squares. We place a vertex of G at the midpoint of each Type-II edge, and a face of G is constructed from the midpoints of four Type-II edges around a square or the midpoints of four Type-II edges around an octagon, see Figure 2.

If a face is bounded by four midpoints of Type-II edges around a square (resp. octagon), we color the face by black (resp. white). It is not hard to see that this way black faces are incident only to white faces and vice versa.

The perfect matchings on the square-octagon lattice, restricted to configurations on Type-II edges, are exactly site percolation configurations on \mathbb{Z}^2 in Ω' . Namely, a Type-II edge is present in a perfect matching of the square-octagon lattice if and only if the vertex of \mathbb{Z}^2 at the midpoint of the edge has state “1” in the corresponding constrained site configuration in Ω' .

For all the Type-II edges we assign weight 1, and for all the horizontal (resp. vertical) Type-I edges we assign weight a (resp. b). Assume $a, b \in (0, \infty)$, and satisfy the following identity

$$(3) \quad a^2 + b^2 = 1.$$

Note that the edge weights a, b satisfying (3) are critical edge weights for the dimer model on the square-octagon lattice, in the sense that a, b satisfying (3) are the only positive a, b such that the corresponding spectral curve intersects the unit torus at a unique real point; for all the other positive a, b the spectral curve does not intersect the unit torus. See [22, 28] for studies of spectral curves for dimer models. Moreover, a, b satisfying (3) are the only positive a, b such that the edge-edge correlation decays polynomially; for all the other positive a, b the edge-edge correlation decays exponentially, see [23]; where the edge-edge correlation is taken with respect to the weak limit of Gibbs measures using torus approximation, see the next paragraph for the construction of such a measure. The dimer model on the square-octagon lattice with critical weights $a, b > 0$ satisfying (3) also corresponds to the free fermion six-vertex model, see [11, 3].

Let S_n be the square-octagon lattice embedded into a $2n \times 2n$ torus; i.e. the vertex set of the corresponding square grid is $\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}$. Let μ_n^a be the Gibbs measure of dimer configurations on S_n , corresponding to weight 1 on Type-II edges, and weight $a, \sqrt{1-a^2}$ on horizontal and vertical Type-I edges, respectively. More precisely, let \mathcal{M}_n be the set of all perfect matchings on S_n , and let $M \in \mathcal{M}_n$ be dimer configuration, then

$$\mu_n^a(M) = \frac{\prod_{e \in M} w_e}{\sum_{M \in \mathcal{M}_n} \prod_{e \in M} w_e},$$

where w_e is the weight of the edge e .

Under (3), μ_n^a satisfies the symmetric condition in Assumption 1.2. As $n \rightarrow \infty$, μ_n^a converges weakly to a translation-invariant measure μ^a (see [23]). Moreover, we have the following lemma

Lemma 2.1. μ^a is $2\mathbb{Z} \times 2\mathbb{Z}$ -ergodic.

Proof. Let \mathcal{R} be the set of all events depending only on dimer configurations of finitely many Type-II edges of G . We consider a σ -algebra \mathcal{F}_2 on dimer configurations of G generated by \mathcal{R} . Let $E_1, E_2 \in \mathcal{R}$, and let $T_{2,h}$ and $T_{2,v}$ be translations by 2 units along horizontal and vertical directions, respectively, then we have

$$(4) \quad \lim_{n \rightarrow \infty} \mu^a(E_1 \cap T_{2,x}^n E_2) = \mu^a(E_1) \mu^a(E_2),$$

where $x \in \{h, v\}$. See [23].

We need to show that all pairs of events satisfying (4) include $\mathcal{F}_2 \times \mathcal{F}_2$. Obviously \mathcal{R} is a π -system since $\emptyset \in \mathcal{R}$ and \mathcal{R} is closed under finite intersections. It suffices to show that all the pairs of events satisfying (4) is a λ -system $\times \lambda$ -system, see [5].

In fact, let (A_1, B) and (A_2, B) be two pairs of events satisfying (4) and $A_2 \subseteq A_1$, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu^a([A_1 \setminus A_2] \cap T_{2,x}^n B) &= \lim_{n \rightarrow \infty} \mu^a[(A_1 \cap T_{2,x}^n B) \setminus (A_2 \cap T_{2,x}^n B)] \\ &= \lim_{n \rightarrow \infty} \mu^a[A_1 \cap T_{2,x}^n B] - \mu^a[A_2 \cap T_{2,x}^n B] \\ &= [\mu^a(A_1) - \mu^a(A_2)] \mu^{\sqrt{2}}(B) \\ &= \mu^a(A_1 \setminus A_2) \mu^a(B), \end{aligned}$$

where the third line follows from the assumption that (A_1, B) and (A_2, B) satisfy (4).

Now let $\{A_n\}_{n=1}^\infty$, $A_n \subseteq A_{n+1}$ be an increasing sequence such that each pair (A_n, B) satisfies (4). Let $D_n = A_n \setminus A_{n+1}$, then by the argument above, we have each (D_n, B) satisfies (4). Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu^a[(\cup_{k=1}^\infty A_k) \cap T_{2,x}^n B] &= \lim_{n \rightarrow \infty} \mu^a[(\cup_{k=1}^\infty D_k) \cap T_{2,x}^n B] \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^\infty \mu^a(D_k \cap T_{2,x}^n B) \end{aligned}$$

Since $\sum_{k=1}^\infty \mu^a(D_k) \leq 1$, there exists $K > 0$, such that

$$\sum_{k=K+1}^\infty \mu^a(D_k \cap T_{2,x}^n B) \leq \sum_{k=K+1}^\infty \mu^a(D_k) < \epsilon.$$

Then

$$\left| \lim_{n \rightarrow \infty} \sum_{k=1}^\infty \mu^a(D_k \cap T_{2,x}^n B) - \lim_{n \rightarrow \infty} \sum_{k=1}^K \mu^a(D_k \cap T_{2,x}^n B) \right| < \epsilon$$

Moreover,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^K \mu^a(D_k \cap T_{2,x}^n B) = \sum_{k=1}^K \mu^a(D_k) \mu^a(B),$$

and

$$\left| \sum_{k=1}^K \mu^a(D_k) \mu^a(B) - \sum_{k=1}^\infty \mu^a(D_k) \mu^a(B) \right| = \sum_{k=K+1}^\infty \mu^a(D_k) < \epsilon,$$

then we have

$$\lim_{n \rightarrow \infty} \mu^a[(\cup_{k=1}^\infty A_k) \cap T_{2,x}^n B] = \sum_{k=1}^\infty \mu^a(D_k) \mu^a(B) = \mu^a(\cup_{k=1}^\infty A_k) \mu^a(B).$$

Then all pairs satisfying (4) is a λ -system \times λ -system including $\mathcal{R} \times \mathcal{R}$, and $\mathcal{R} \times \mathcal{R}$ is a π -system. Hence for all pairs in $\mathcal{F}_2 \times \mathcal{F}_2$, (4) is true. We conclude that μ is $2\mathbb{Z} \times 2\mathbb{Z}$ ergodic, because it is $2\mathbb{Z} \times 2\mathbb{Z}$ translation-invariant and mixing (see [14]). □

We construct an Ising model with spins located on vertices of the dual square grid \mathbb{L}^* . Assume all the horizontal edges of \mathbb{L}^* have coupling constant J_h , and all the vertical

edges have coupling constant J_v , such that $J_h, J_v \in (0, \infty)$, and J_h, J_v can be computed according to the following formulae:

$$(5) \quad a = \frac{2e^{-2J_v}}{1 + e^{-4J_v}} = \frac{1 - e^{-4J_h}}{1 + e^{-4J_h}}$$

$$(6) \quad b = \frac{2e^{-2J_h}}{1 + e^{-4J_h}} = \frac{1 - e^{-4J_v}}{1 + e^{-4J_v}}$$

Equations (5) and (6) are solvable given (3). Moreover, the corresponding J_h, J_v obtained by solving (3), (5), (6) are coupling constants for the critical Ising model.

The XOR Ising model (see [35]) is a spin configuration on vertices of \mathbb{L}^* , such that for any $v \in V(\mathbb{L}^*)$, we have

$$\sigma_{XOR}(v) = \sigma_1(v)\sigma_2(v),$$

where σ_1, σ_2 are two independent, identically distributed Ising models on vertices of \mathbb{L}^* . Assume both σ_1 and σ_2 have coupling constants given by (5), (6).

We say an XOR Ising model is defined on \mathbb{L} (resp. \mathbb{L}^*) if the XOR Ising model has spins located on vertices of \mathbb{L} (resp. \mathbb{L}^*), and interactions between nearest neighboring vertices of \mathbb{L} (resp. \mathbb{L}^*).

The monochromatic contours for an XOR Ising configuration, σ_{XOR} , defined on \mathbb{L} (resp. \mathbb{L}^*) are contours consisting of edges of \mathbb{L}^* (resp. \mathbb{L}), separating neighboring vertices of \mathbb{L} (resp. \mathbb{L}^*) with different states of σ_{XOR} . The monochromatic contours of the XOR Ising model was first studied in [35], in which the scaling limits of monochromatic contours of the critical XOR Ising model are conjectured to be level lines of Gaussian free field. It is proved in [6] that the monochromatic contours of the XOR Ising model on the square grid correspond to level lines of height functions of the dimer model on the square-octagon lattice, inspired by the correspondence between Ising model and bipartite dimer model in [10]. We will study the percolation properties of the XOR Ising model, as an application of the main theorems proved in this paper for the general constrained percolation process.

Theorem 2.2. *For any $0 < a < 1$, μ^a -a.s. there are no infinite present Type-II clusters or infinite absent Type-II clusters.*

Proof. As discussed before, any dimer configuration on the square-octagon lattice corresponds to a contour configuration on $\mathbb{L} \cup \mathbb{L}^*$, such that each vertex in \mathbb{L} and \mathbb{L}^* has an even number of incident present edges, and the primal contours and the dual contours can never intersect.

After removing the dual contours, the primal contours have the same distribution as monochromatic contours of the XOR-Ising model, with spins located at vertices of the dual lattice \mathbb{L}^* , if the parameters of the two i.i.d. Ising models satisfy (5) and (6), see [6]. Similarly, removing the primal contours, the dual contours correspond to monochromatic contours of the critical XOR-Ising model with spins located at vertices of the primal lattice \mathbb{L} .

The corresponding probability measure for the XOR-Ising model is translation-invariant and satisfies the finite-energy property, then the theorem follows from Theorem 1.5. \square

Theorem 2.3. *The critical XOR-Ising model, which is the product of two independent critical Ising models with the same coupling constants, almost surely has no infinite “+”-clusters, and no infinite “-”-clusters, under the unique infinite-volume Gibbs measure.*

Proof. First of all, we know that the spontaneous magnetization for the critical Ising model vanishes (see [9]), this implies that the critical Ising model has a unique infinite-volume Gibbs measure (see [1]). Therefore the critical XOR-Ising model, as the product of two independent critical Ising models with the same coupling constant, has a unique infinite-volume Gibbs measure. It suffice to show the almost sure nonexistence of infinite “+”-clusters and infinite “-”-clusters under the weak limit of measures using the torus approximation. Since the infinite “+” or “-” clusters correspond to the infinite components bounded by dual contours, and almost surely there are no infinite components bounded by dual contours, by Theorem 1.6, we conclude that almost surely there are no infinite “+”-clusters, or infinite “-”-clusters in the critical XOR-Ising model. \square

Now let us turn to the non-critical XOR Ising model. Let σ_1, σ_2 be two i.i.d Ising spin configurations defined on vertices of the dual lattice \mathbb{L}^* , with non-critical coupling constants J_h and J_v on horizontal and vertical edges, respectively. As before, the spin of the XOR Ising model is a product of σ_1 and σ_2 . According to [6], such a non-critical XOR Ising model also corresponds to a dimer model on the square-octagon lattice, in which each Type-II edge has weight 1, each Type-I edge parallel to an edge e of \mathbb{L}^* has weight $\frac{1-e^{-4J_e}}{1+e^{-4J_e}}$, and each Type-I edge perpendicular to an edge e of \mathbb{L}^* has weight $\frac{2e^{-2J_e}}{1+e^{-4J_e}}$. The monochromatic contours of the XOR Ising model correspond to primal contours of the perfect matching. Note that the weak limit of Gibbs measures for dimer configurations on the square-octagon lattice, corresponding to the weights of the non-critical XOR Ising model, using torus approximation, is no longer $\mathbb{Z} \times \mathbb{Z}$ translation invariant, but $2\mathbb{Z} \times 2\mathbb{Z}$ translation invariant.

Theorem 2.4. *The non-critical XOR Ising model has at most one infinite monochromatic contour, under any translation-invariant Gibbs measure.*

Proof. For the non-critical XOR Ising model, the distribution of monochromatic contours is unique under any translation-invariant Gibbs measure, because the distribution of interfaces is unique under any translation-invariant Gibbs measure for a non-critical Ising model, see [26]. Let ν be the distribution of monochromatic contours under a translation-invariant Gibbs measure. Then ν is the marginal distribution for primal contours of a $2\mathbb{Z} \times 2\mathbb{Z}$ translation-invariant, $2\mathbb{Z} \times 2\mathbb{Z}$ ergodic Gibbs measure on the space Ω' of the constrained percolation process. Then the theorem follows from Theorem 1.7. \square

The unique infinite contour can be proved unstable (almost surely does not exist), if the underlying Gibbs measure is translation-invariant, and satisfies the F.K.G. lattice condition (see [1, 33]). In the following theorem, we prove the nonexistence of the unique infinite contour without the F.K.G. lattice condition.

Theorem 2.5. *Under any translation-invariant Gibbs measure, almost surely the low-temperature XOR Ising model has no infinite monochromatic contours.*

Proof. Let σ_1, σ_2 be two i.i.d. low temperature Ising configurations, with spins placed on vertices of the dual lattice \mathbb{L}^* . Assume each horizontal (resp. vertical) edge of \mathbb{L}^* has coupling constant J_h (resp. J_v). The Ising model is in the low temperature state if and only if the coupling constant satisfies

$$(7) \quad e^{-2J_h} + e^{-2J_v} + e^{-2(J_h+J_v)} < 1.$$

We define another Ising model on the vertices of the primal lattice \mathbb{L} , such that each horizontal edge of \mathbb{L} has coupling constant J'_h (resp. J'_v), satisfying

$$(8) \quad e^{-2J'_h} = \frac{1 - e^{-2J_v}}{1 + e^{-2J_v}},$$

$$(9) \quad e^{-2J'_v} = \frac{1 - e^{-2J_h}}{1 + e^{-2J_h}}.$$

Given (7), (8), (9), we have

$$(10) \quad e^{-2J'_h} + e^{-2J'_v} + e^{-2(J'_h+J'_v)} > 1.$$

(10) implies that this Ising model on \mathbb{L} is in the high temperature state.

It is known that the high-temperature Ising model has a unique infinite-volume Gibbs measure (see [1]), and this Gibbs measure is translation-invariant; therefore the high-temperature XOR Ising model, each spin of which is the product of two i.i.d. high-temperature Ising spins, has a unique infinite-volume Gibbs measure π , and this Gibbs measure is translation-invariant, and moreover, ergodic.

In fact, the low-temperature XOR Ising model on \mathbb{L}^* , and the high-temperature XOR Ising model on \mathbb{L} , with coupling constants satisfying (8), (9), correspond to dimer models on the square-octagon lattice with the same edge weights. More precisely, each Type-II edge has weight 1. Each Type-I edge parallel to an edge e of \mathbb{L}^* has weight $\frac{1-e^{-4J_e}}{1+e^{-4J_e}}$ in the dimer model corresponding to the low-temperature XOR Ising model on \mathbb{L}^* , according to the weight correspondence described in the paragraph before Theorem 2.4. Moreover, this Type-I edge must be perpendicular to an edge e_* of \mathbb{L} , and it has weight $\frac{2e^{-2J_{e_*}}}{1+e^{-4J_{e_*}}}$, in the dimer model corresponding to the high-temperature XOR Ising model on \mathbb{L} . But $\frac{1-e^{-4J_e}}{1+e^{-4J_e}} = \frac{2e^{-2J_{e_*}}}{1+e^{-4J_{e_*}}}$, if J_e and J_{e_*} satisfy (8), (9). Similarly, each Type-I edge parallel to an edge of \mathbb{L} and perpendicular to an edge of \mathbb{L}^* has the same edge weight in the dimer model corresponding to the high-temperature XOR Ising model on \mathbb{L} , and in the dimer model corresponding to the low-temperature XOR Ising model on \mathbb{L}^* .

The distribution of monochromatic contours for the low-temperature XOR Ising model on \mathbb{L}^* (which is unique under any infinite-volume, translation-invariant Gibbs measure), is the same as the distribution of the primal contours of the dimer model of the square-octagon lattice, under the infinite volume Gibbs measure θ , obtained from the torus approximation.

Let P_1 be the event that there is a unique infinite primal contour. Using the same argument as in Lemma 2.1, we can prove that θ is $2\mathbb{Z} \times 2\mathbb{Z}$ ergodic. Hence either $\theta(P_1) = 0$, or $\theta(P_1) = 1$.

If $\theta(P_1) = 1$, then in the high-temperature XOR Ising model on \mathbb{L} , almost surely there is an infinite cluster π containing the infinite primal contour. Let E_+ (resp. E_-) be the event that there exists an infinite $+$ -cluster (resp. $-$ -cluster) containing an infinite primal contour, in the high-temperature XOR Ising model on \mathbb{L} , then $\pi(E_+ \cup E_-) = 1$

implies $\pi(E_+ \cap E_-) = 1$, by ergodicity and symmetry of π , and by translation invariance of E_+ and E_- . Then π -a.s. there are at least two infinite clusters containing two distinct primal contours, hence θ -a.s. there are at least two infinite primal contours. But this is a contradiction to Theorem 2.4. Hence $\theta(P_1) = 0$, and the theorem follows from Theorem 2.4. \square

3. CONTOURS AND CLUSTERS

In this section, we prove a few combinatorial and probabilistic results regarding infinite contours and infinite clusters in the constrained percolation process, in preparation for the proof of Theorems 1.4-1.7. Let μ be a probability measure on Ω' satisfying Assumptions 1.1 and 1.2. We begin with the following elementary lemma.

Lemma 3.1. *Let S be a primal or a dual face in \mathbb{L} or \mathbb{L}^* . $S \cap \mathbb{Z}^2$ consists of four vertices. If in a constrained configuration in Ω' , all the four vertices in $S \cap \mathbb{Z}^2$ have state “0”, then flipping the states of all vertices in S to “1”, and preserve the states of all the other vertices of \mathbb{Z}^2 , we still obtain a constrained configuration in Ω' , and vice versa.*

Proof. As explained before, the constrained configurations in Ω' correspond to contour configurations in Φ' . To prove that after the state change in the way described in the theorem, we still have a constrained site configuration in Ω' , it suffices to check that after the state change, the new configuration still corresponds to a contour configuration in Ω' .

Changing states for all the four vertices inside a single square S corresponds to changing states on the four boundary edges of the square, i.e. present edges become absent, and absent edges become present. For example, let $e \in E(\mathbb{L}) \cup E(\mathbb{L}^*)$ be a boundary edge of S , let v be a vertex in $\mathbb{Z}^2 \setminus S$, so that v is incident to a vertex in S , through an edge of G crossing e . Assume before the state change, e is present in the contour configuration, i.e. before the state change, in the site configuration of \mathbb{Z}^2 , all the four vertices in $S \cap \mathbb{Z}^2$ have state “0”, v has state “1”. If we change the states of all vertices in S to “1”, and preserve the states of all other vertices in \mathbb{Z}^2 , then in the changed contour configuration, e must be absent.

Since before the change, in the contour configuration, each vertex of \mathbb{L} or \mathbb{L}^* has an even number of incident present edges, after the change in the new contour configuration, each vertex still has an even number of incident present edges, because we change states on exactly two incident edges for each one of the four vertices of the square; and for any other vertex of \mathbb{Z}^2 , we do not change the states of its incident edges.

Moreover, since before the change, all the vertices in the square S have the same state, no present edges in the corresponding contour configuration intersect the squares. Therefore after the change of states, no present edges of \mathbb{L} or \mathbb{L}^* intersect the boundary of the square either, because after the change, all the vertices inside the square S still have the same state and there are no present edges of \mathbb{L} or \mathbb{L}^* intersecting the boundary of S to separate the state “0” and the state “1”.

Then we conclude that after the change of states in the way as described in the theorem, we still have a constrained configuration in Ω' . \square

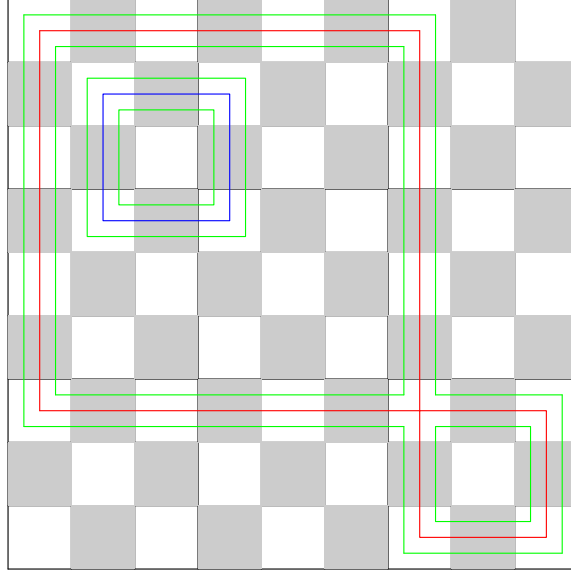


FIGURE 3. Primal Contour, Dual Contour and Interface

We consider an augmented square grid $A\mathbb{Z}^2$, whose vertices are either vertices of \mathbb{Z}^2 , centers of faces of \mathbb{Z}^2 , or midpoints of edges of \mathbb{Z}^2 . Two vertices u, w of $A\mathbb{Z}^2$ are connected by an edge of $A\mathbb{Z}^2$ if and only if one of the following cases occurs:

- I u is a vertex of \mathbb{Z}^2 , and w is the midpoint of an edge of \mathbb{Z}^2 , and u is an endpoint of the edge corresponding to w , or vice versa.
- II u is the center of a face of \mathbb{Z}^2 , and w is the midpoint of an edge of \mathbb{Z}^2 , and the edge corresponding to w is on the boundary of the face corresponding to u , or vice versa.

Note that $A\mathbb{Z}^2$ is a square grid with edge length $\frac{1}{2}$. Let $[A\mathbb{Z}^2]^*$ be the dual lattice of $A\mathbb{Z}^2$.

Let $\phi \in \Phi'$ be a contour configuration. To each contour C in ϕ , we associate a union of disjoint cycles or infinite paths in $[A\mathbb{Z}^2]^*$ called the interface of the contour. More precisely, we consider a “thickened” version of C , \tilde{C} , by replacing each present edge of C with a bar of width $\frac{1}{2}$, such that each present edge of C is located in the center of the corresponding bar. Note that $\mathbb{R}^2 \setminus \tilde{C}$ is a union of disjoint connected components. The boundaries of components of $\mathbb{R}^2 \setminus \tilde{C}$ form the interface of the contour C . In particular, each component of the interface of the contour C is either a cycle or a infinite self-avoiding path, consisting of edges of $[A\mathbb{Z}^2]^*$, such that each vertex of $[A\mathbb{Z}^2]^*$ is incident to 0 or 2 present edges. Here by cycle we mean a finite connected component of edges of $[A\mathbb{Z}^2]^*$ in which each vertex of $[A\mathbb{Z}^2]^*$ has two incident edges.

The interface of a contour configuration $\phi \in \Phi'$ is the union of interfaces of all contours in ϕ . Again the interface of ϕ is a set of disjoint cycles or infinite paths in $[A\mathbb{Z}^2]^*$. See Figure 3 for an example of the interface, where red and blue lines are primal and dual contours, and green lines are interface, see also [15].

Lemma 3.2. *If μ -a.s. there exist two infinite contours, then μ -a.s. there exists an infinite (“0” or “1”)-cluster.*

Proof. If μ -a.s. there exist two infinite contours, then we can find two distinct infinite contours C_1 and C_2 , two points $x \in C_1$ and $y \in C_2$, a self-avoiding path p_{xy} , consisting of edges of G and two half-edges, one starting at x and the other ending at y , and connecting x and y , such that p_{xy} does not intersect any infinite contours except at x and at y .

Let $v \in \mathbb{Z}^2$ be the first vertex along p_{xy} starting from x . Let u be the midpoint of the line segment $[v, x]$. The interface of C_1 passes u . Let ℓ_u be the connected component of the interface of C_1 passing u . Then ℓ_u is either an infinite self-avoiding path or a cycle consisting of edges of $[A\mathbb{Z}^2]^*$, such that along ℓ_u each vertex of $[A\mathbb{Z}^2]^*$ is incident 2 present edges of $[A\mathbb{Z}^2]$ in the interface.

If ℓ_u is an infinite self-avoiding path, then we claim that v is in an infinite (“0” or “1”)-cluster of the constrained site configurations on \mathbb{Z}^2 . To see why this is true, note that $\mathbb{R}^2 \setminus \ell_u$ has two components, Q_v and Q_v^c , where Q_v is the component including v . Find all the vertices in $Q_v \cap \mathbb{Z}^2$ satisfying one of the following conditions

- I the distance of the vertex to ℓ_u is $\frac{1}{4}$;
- II the distance of the vertex to ℓ_u is $\frac{\sqrt{2}}{4}$, and the vertex is incident to two vertices in $Q_v \cap \mathbb{Z}^2$ whose distances to ℓ_u are $\frac{1}{4}$.

All the vertices satisfying one of the above conditions form an infinite, connected set V_I , in which every vertex has the same state. Then there is an infinite cluster of the constrained site configuration in Ω' including all the vertices in V_I .

If ℓ_u is a cycle, then $\mathbb{R}^2 \setminus \ell_u$ has two components, Q_v and Q_v^c , where Q_v is the component including v . Since ℓ_u is a cycle, exactly one of Q_v and Q_v^c is finite, the other is infinite. Since $C_1 \subseteq Q_v^c$, and C_1 is an infinite contour, we deduce that Q_v^c is infinite, and Q_v is finite. Since $y \notin \ell_u$, either $y \in Q_v$, or $y \in Q_v^c$. If $y \in Q_v^c$, then any path connecting v and y must intersect C_1 . In particular, p_{xy} intersects C_1 not only at x , but also at some point other than x . This is a contradiction. Hence $y \in Q_v$. Since $C_1 \cap C_2 = \emptyset$, this implies $C_2 \subseteq Q_v$; because if $C_2 \cap Q_v^c \neq \emptyset$, then $C_2 \cap C_1 \neq \emptyset$. But $C_2 \subseteq Q_v$ is impossible since C_2 is infinite and Q_v is finite. Hence ℓ_u must be an infinite self-avoiding path.

Therefore we conclude that if μ -a.s. there exist two infinite contours, then μ -a.s. there exists an infinite (“0” or “1”)-cluster. □

Lemma 3.2 has the following straightforward corollary.

Corollary 3.3. *If μ -a.s. there are no infinite clusters, then μ -a.s. there are no infinite contours.*

Proof. Let \mathcal{A}_1 (resp. \mathcal{A}_2) be the event that there exist infinite primal (resp. dual) contours. There is a 1-to-1 correspondence between configurations in \mathcal{A}_1 and configurations in \mathcal{A}_2 . Namely, we translate each configuration in \mathcal{A}_1 by $(1, 1)$, and obtain a configuration in \mathcal{A}_2 , and vice versa. By translation invariance of μ , we have $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2)$.

Moreover, since \mathcal{A}_1 and \mathcal{A}_2 are $2\mathbb{Z} \times 2\mathbb{Z}$ translation invariant events, by ergodicity of μ , we have either $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) = 0$, or $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) = 1$.

If $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) = 1$, since primal contours and dual contours can never intersect, μ -a.s. there exist at least two distinct infinite contours. by Lemma 3.2, μ -a.s. there exists an infinite cluster. This contradiction implies the corollary. \square

Lemma 3.4. *Let C_∞ be an infinite contour, then there is an infinite cluster adjacent to C_∞ in each infinite component of $\mathbb{R}^2 \setminus C_\infty$.*

Proof. Let S be an infinite component of $\mathbb{R}^2 \setminus C_\infty$, and I_S be the interface of C_∞ in S , then I_S is infinite and connected. Because if I_S is finite, then either S or C_∞ is finite, but this is impossible. I_S is connected, because S is simply-connected. Consider all the vertices of \mathbb{Z}^2 satisfying one of the following conditions:

- $v \in S \cap \mathbb{Z}^2$, and $\text{dist}(v, I_S) = \frac{1}{4}$,
- $v \in S \cap \mathbb{Z}^2$, and $\text{dist}(v, I_S) = \frac{\sqrt{2}}{4}$, and v is adjacent to two vertices in $S \cap \mathbb{Z}^2$ whose distances to I_S are $\frac{1}{4}$.

Then all the vertices satisfying one of the above conditions form part of an infinite cluster, which is adjacent to C_∞ . \square

4. NONEXISTENCE OF THE UNIQUE INFINITE “1”-CLUSTER

In this section, we prove Theorem 1.4. Let μ be a probability measure on Ω' satisfying Assumptions 1.1 and 1.2.

Lemma 4.1. *Assume there is exactly one infinite “0”-cluster and exactly one infinite “1”-cluster. Let $x \in \mathbb{Z}^2$ be in the infinite “0”-cluster, $y \in \mathbb{Z}^2$ be in the infinite “1”-cluster, and ℓ_{xy} be a path, consisting of edges of G and connecting x and y , then ℓ_{xy} intersects infinite contours an odd number of times.*

Proof. Moving along ℓ_{xy} , two neighboring vertices u and v in $\mathbb{Z}^2 \cap \ell_{xy}$ have different states if and only if the edge $\langle u, v \rangle$ intersects a contour. Since the states of x and y are different, moving along ℓ_{xy} , the states of vertices must change an odd number of times. Therefore ℓ_{xy} intersects (primal and dual) contours an odd number of times.

Since ℓ_{xy} intersects finitely many finite contours in total, let C_{f_1}, \dots, C_{f_m} be all the finite contours intersecting ℓ_{xy} , where $m = 0, 1, 2, \dots$. We make the convention here that $m = 0$ means ℓ_{xy} does not intersect finite contours at all, and $\cup_{i=1}^m C_{f_i} = \emptyset$.

Let \mathbb{R}^2 be the plane. We claim that $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$ has exactly one infinite component. To see why this is true, we consider the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$. Since C_{f_i} is a finite contour, we have, for $1 \leq i \leq m$, $\mathbb{R}^2 \cup \{\infty\} \setminus C_{f_i}$ includes a neighborhood of ∞ . Then there is a unique (open) component of $\mathbb{R}^2 \cup \{\infty\} \setminus \cup_{i=1}^m C_{f_i}$, including a neighborhood of the point ∞ , denoted by U_∞ . We can see that $U_\infty \setminus \{\infty\}$ is still connected, that is because, for $a, b \in U_\infty \setminus \{\infty\}$, there exists a path $\ell_{ab} \subseteq U_\infty$ connecting a and b . Since U_∞ includes a neighborhood of ∞ , if $\infty \in \ell_{ab}$, a small perturbation of ℓ_{ab} generates a path in $U_\infty \setminus \{\infty\}$ connecting a and b . Then $U_\infty \setminus \{\infty\}$ is the unique infinite component of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$.

Then both x and y lie in the infinite connected component of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$. That is because if x is in a finite component of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$, then it is a contradiction to the fact that x is in an infinite “0”-cluster, because the infinite “0”-cluster including x cannot be bounded in a finite component of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$. Similarly y is also in an infinite component

of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$. Since $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$ has a unique infinite component, we infer that both x and y are in the same infinite component of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$.

Since both x and y lie in the infinite connected component of $\mathbb{R}^2 \setminus \cup_{i=1}^m C_{f_i}$, we can find a path ℓ'_{xy} connecting x and y , using edges of G , such that the path does not intersect $\cup_{i=1}^m C_{f_i}$ at all. Moreover, each vertex of $\mathbb{L} \cup \mathbb{L}^*$ has an even number of present edges in $\cup_{i=1}^m C_{f_i}$, we infer that ℓ_{xy} intersects $\cup_{i=1}^m C_{f_i}$ an even number of times, because changing paths from ℓ'_{xy} to ℓ_{xy} does not change the parity of intersections of the path with $\cup_{i=1}^m C_{f_i}$. This implies that ℓ_{xy} must intersect infinite contours an odd number of times, because ℓ_{xy} intersects (infinite and finite) contours an odd number of times in total, and ℓ_{xy} intersects finite contours an even number of times. □

Lemma 4.2. *Assume there is exactly one infinite “0”-cluster and exactly one infinite “1”-cluster. Assume there exist a vertex x in the infinite “0”-cluster, a vertex y in the infinite “1”-cluster, and a path ℓ_{xy} , consisting of edges of G and connecting x and y , such that ℓ_{xy} intersects exactly one infinite contour, C_∞ , then C_∞ is adjacent to both the infinite “0”-cluster and the infinite “1”-cluster.*

Proof. By lemma 3.4, there is an infinite cluster in each infinite component of $\mathbb{R}^2 \setminus C_\infty$. Since there are exactly two infinite clusters, $\mathbb{R}^2 \setminus C_\infty$ has at most two infinite components.

If $\mathbb{R}^2 \setminus C_\infty$ has no infinite components, this is impossible, since in this case there will be no infinite clusters.

If $\mathbb{R}^2 \setminus C_\infty$ has exactly two infinite components, then we can construct two infinite connected set of vertices in the two infinite components of $\mathbb{R}^2 \setminus C_\infty$, as in Lemma 1.4, denoted by V_1 and V_2 , such that C_∞ is adjacent to both V_1 and V_2 . Moreover, V_1 and V_2 are exactly part of the infinite “0”-cluster and part of the infinite “1”-cluster. Therefore C_∞ is adjacent to both the infinite “0”-cluster and the infinite “1”-cluster.

If $\mathbb{R}^2 \setminus C_\infty$ has exactly one infinite component, denoted by R_{C_∞} , then both the infinite “0”-cluster and the infinite “1”-cluster lie in R_{C_∞} , in particular $x, y \in R_{C_\infty}$. We can find a path $\tilde{\ell}_{xy}$, connecting x and y , using edges of G , such that $\tilde{\ell}_{xy}$ does not intersect C_∞ at all. Changing path from $\tilde{\ell}_{xy}$ to ℓ_{xy} does not change the parity of the number of intersections of the path with C_∞ , we infer that ℓ_{xy} intersects C_∞ an even number of times. But this is a contradiction, since ℓ_{xy} intersects C_∞ an odd number of times, by Lemma 4.1. □

Lemma 4.3. *If there is exactly one infinite “0”-cluster and exactly one infinite “1”-cluster, then there exists an infinite primal or dual contour, adjacent to both the infinite “0”-cluster and the infinite “1”-cluster.*

Proof. Let x be a vertex in the infinite “0”-cluster, and let y be a vertex in the infinite “1” cluster. Let ℓ_{xy} be a path consisting of edges of the original square grid G connecting x and y .

By Lemma 4.1, ℓ_{xy} must intersect infinite contours an odd number of times. By lemma 4.2, if ℓ_{xy} intersect exactly one infinite contour, C_∞ , then C_∞ is adjacent to both the infinite “0”-cluster and the infinite “1”-cluster, then the lemma is proved in the case when ℓ_{xy} intersects exactly one infinite contour.

If there exist more than one infinite contours intersecting ℓ_{xy} , let C_1 and C_2 be two distinct infinite contours intersecting ℓ_{xy} .

Let $u \in C_1 \cap \ell_{xy}$ and $v \in C_2 \cap \ell_{xy}$, such that the portion of ℓ_{xy} between u and v , p_{uv} , does not intersect any infinite contours except at u and at v . As in the proof of Lemma 3.2, let u_1 be the first vertex of \mathbb{Z}^2 along p_{uv} , starting from u ; and let v_1 be the first vertex of \mathbb{Z}^2 along p_{uv} starting from v . Let u_2 (resp. v_2) be the midpoint of the line segment $[u, u_1]$ (resp. $[v, v_1]$). According to the proof of Lemma 3.2, the component of the interface of C_1 passing u_2 is infinite, denoted by I_1 , and the component of the interface of C_2 passing v_2 is infinite, denoted by I_2 . Following the procedure in the proof of Lemma 3.2, we can find an infinite cluster ξ_1 along I_1 , such that $u_1 \in \xi_1$. The following cases might happen

- I $x \notin \xi_1$, and $y \notin \xi_1$;
- II $x \notin \xi_1$, and $y \in \xi_1$;
- III $x \in \xi_1$, and $y \notin \xi_1$;
- IV $x \in \xi_1$, and $y \in \xi_1$.

First of all, Case IV is impossible because we assume x and y are in two distinct infinite clusters. Secondly, if Case I is true, then there exist at least 3 infinite clusters, this is a contradiction to the assumption that there exists exactly one infinite “0”-cluster and one infinite “1”-cluster.

Case II and Case III can be proved using similar arguments, and we write down the proof of Case II here.

If Case II is true, first note that $y \in \xi_1$ implies that C_1 is adjacent to the infinite “1”-cluster. Let z be the first point in $C_1 \cap \ell_{xy}$, when traveling along ℓ_{xy} starting from x . Let p_{xz} be the portion of ℓ_{xy} between x and z .

Next, we will prove the following claim.

Claim 4.4. *Under Case II, there is an infinite contour adjacent to both the infinite “0”-cluster and the infinite “1”-cluster.*

We will prove Claim 4.4 by an induction on the length of p_{xz} . Note that the length of p_{xz} is always of the form $n + \frac{1}{2}$, where $n = 0, 1, 2, \dots$.

First of all, when $n = 0$, the length of p_{xz} is $\frac{1}{2}$. This implies that C_1 is adjacent to the infinite “0”-cluster at x . Under Case II, C_1 is also adjacent to the infinite “1”-cluster at y , then Claim 4.4 is proved.

We make the following induction hypothesis

- Assume Claim 4.4 holds for $n \leq k$ ($k \geq 0$).

Now we consider the case when $n = k + 1$. The interior points of p_{xz} are all points along p_{xz} except x and z . We claim that if at interior points, p_{xz} intersects only finite contours, then C_1 is adjacent to both the infinite “0”-cluster and the infinite “1”-cluster. Under Case II, it suffices to show that C_1 is adjacent to the infinite “0”-cluster.

Since p_{xz} intersects finitely many finite contours at interior points, let C_{f_1}, \dots, C_{f_r} be all the finite contours intersecting p_{xz} . Using the same argument as in the proof of Lemma 4.1, we infer that $\mathbb{R}^2 \setminus \cup_{i=1}^r C_{f_i}$ has a unique infinite component, and both x and z are in the infinite component of $\mathbb{R}^2 \setminus \cup_{i=1}^r C_{f_i}$. That is because x is in the infinite “0”-cluster, and z is in the infinite contour C_1 ; neither the infinite “0”-cluster nor the infinite contour C_1 can be bounded in a finite component of $\mathbb{R}^2 \setminus \cup_{i=1}^r C_{f_i}$.

Let I_f be the interface of $\cup_{i=1}^r C_{f_i}$ in the unique infinite cluster of $\mathbb{R}^2 \setminus \cup_{i=1}^r C_{f_i}$. Since each C_{f_i} , $1 \leq i \leq r$, is a finite contour, each component of the interface of C_{f_i} is finite. In particular, I_f consists of finitely many disjoint cycles, denoted by D_1, \dots, D_t . For $1 \leq i \leq t$, $\mathbb{R}^2 \setminus D_i$ has exactly one infinite component, and one finite component. Let B_i be the finite component of $\mathbb{R}^2 \setminus D_i$, define

$$\tilde{B}_i = \{S \text{ is a unit square in } G : S \cap B_i \neq \emptyset\}.$$

In particular, \tilde{B}_i is a closed set consisting of unit squares in G . Define $B = \cup_{i=1}^t \tilde{B}_i$; let B' be the interior of B . Note that B' is open, and $x, z \in \mathbb{R}^2 \setminus B'$, although x may be on the boundary of B' .

There is a path $p'_{xz} \subseteq [p_{xz} \cap (\mathbb{R}^2 \setminus B')] \cup \partial B'$, connecting x and z , where $\partial B'$ is the boundary of B' . More precisely, p_{xz} is divided by $\partial B'$ into segments; on each segment of p_{xz} in $\mathbb{R}^2 \setminus B'$, p'_{xz} follow the path of p_{xz} ; for each segment of p_{xz} in B' , p'_{xz} follow the boundary of B' to connect the two endpoints of the segment. All the vertices of p'_{xz} are in the same cluster; we infer that C_1 is incident to the infinite “0”-cluster at x , if p_{xz} intersects only finite contours at interior points.

Now assume p_{xz} intersects infinite contours at interior points. Let C_3 be an infinite contour intersecting p_{xz} at interior points. Obviously, C_3 and C_1 are distinct, because C_1 intersects p_{xz} only at z . Let w be the last point in $C_3 \cap p_{xz}$, when traveling along p_{xz} , starting from x , and let p_{wz} be the portion of p_{xz} between w and z . Assume p_{wz} does not intersect infinite contours at interior points.

Let w_1 be the first vertex in \mathbb{Z}^2 along p_{wz} , starting from w , and let w_2 be the midpoint of w and w_1 . According to the proof of Lemma 3.2, the component of the interface of C_3 passing w_2 is infinite, denoted by I_3 ; and we can find an infinite cluster ξ_3 along I_3 , such that $w_1 \in \xi_3$. The following cases might happen

- i $x \notin \xi_3$, and $y \notin \xi_3$;
- ii $x \in \xi_3$, and $y \notin \xi_3$;
- iii $x \notin \xi_3$, and $y \in \xi_3$;
- iv $x \in \xi_3$, and $y \in \xi_3$.

First of all, Case iv is impossible because we assume x and y are in two distinct infinite clusters. Secondly, if Case i is true, then there exist at least 3 infinite clusters, this is a contradiction to the assumption that there exists exactly one infinite “0”-cluster and one infinite “1”-cluster.

If Case ii is true, then C_3 is adjacent to the infinite “0”-cluster including x ; and since $w_1 \in \xi_3$, and p_{wz} does not intersect infinite contours except at w and z , therefore $z \in \xi_3$, and ξ_3 is exactly the infinite “0”-cluster including x , we conclude that C_1 is incident to the infinite “0”-cluster including x as well, and Claim 4.4 is proved.

If Case iii is true, then C_3 is adjacent to the infinite “1”-cluster including y . Let t be the first vertex in $p_{xz} \cap C_3$, when traveling from p_{xz} , starting at x , and let p_{xt} be the portion of p_{xz} between x and t . We explore the path p_{xt} as we have done for p_{xz} . Since the length of p_{xz} is finite, and the length of p_{xt} is less than that of p_{xz} by at least 1, we apply the induction hypothesis with C_1 replaced by C_3 , C_2 replaced by C_1 , ξ_1 replaced by ξ_3 , p_{xz} replaced by p_{xt} , and we conclude that there exists an infinite contour adjacent to both the infinite “0”-cluster and infinite “1”-cluster.

□

4.1. Proof of Theorem 1.4. Since \mathcal{A} is a translation-invariant event, by ergodicity of μ , either $\mu(\mathcal{A}) = 1$ or $\mu(\mathcal{A}) = 0$. Assume $\mu(\mathcal{A}) = 1$, let \mathcal{B} be the event that there exists a unique infinite “0”-cluster. Then by symmetry of μ , we have $\mu(\mathcal{B}) = 1$, hence $\mu(\mathcal{A} \cap \mathcal{B}) = 1$.

As explained before the constrained site configurations in Ω' correspond to contour configurations in Φ' .

If μ -a.s. there exists exactly one infinite “0”-cluster and exactly one infinite “1”-cluster simultaneously, then by lemma 4.3, μ -a.s. there exists an infinite primal or dual contour adjacent to both the infinite “0”-cluster and the infinite “1”-cluster. Let \mathcal{D}_1 (resp. \mathcal{D}_2) be the event that there exists an infinite primal (resp. dual) contour in \mathbb{L} (resp. \mathbb{L}^*), adjacent to both the infinite “0”-cluster and the infinite “1”-cluster. Given $\mu(\mathcal{A} \cap \mathcal{B}) = 1$, we have

$$(11) \quad \mu(\mathcal{D}_1 \cup \mathcal{D}_2) = 1.$$

By translation invariance, we have

$$(12) \quad \mu(\mathcal{D}_1) = \mu(\mathcal{D}_2).$$

Moreover, since μ is $2\mathbb{Z} \times 2\mathbb{Z}$ -ergodic, we have either

$$(13) \quad \mu(\mathcal{D}_1) = 0, \text{ or } \mu(\mathcal{D}_1) = 1.$$

Combining (11), (12) and (13), we have

$$\mu(\mathcal{D}_1 \cap \mathcal{D}_2) = 1.$$

More precisely, we have exactly one infinite “1”-cluster on \mathbb{Z}^2 , denoted by C_1 and exactly one infinite “0”-cluster on \mathbb{Z}^2 , denoted by C_0 . There is an infinite primal contour adjacent to both C_0 and C_1 , denoted by S ; as well as an infinite dual contour adjacent to both C_0 and C_1 , denoted by W .

Therefore, we can find points $x \in S$ and $y \in W$, such that x and y are connected by a path ℓ_{xy} , consisting of edges of G and two half-edges, one starts at x and one ends at y , such that every vertex in $\ell_{xy} \cap \mathbb{Z}^2$ is in C_1 . Similarly, we can find a point $p \in S$ and $q \in W$, such that p and q are connected by a path ℓ_{pq} , consisting of edges of G and two half-edges, one starts at x and one ends at y , such that every point of $\ell_{pq} \cap \mathbb{Z}^2$ is in C_0 .

Moreover, we can find a path $\ell_{px} \subseteq S$ connecting p and x and $\ell_{qy} \subseteq W$ connecting q and y . The paths ℓ_{xy} , ℓ_{qy} , ℓ_{pq} and ℓ_{px} form a closed cycle bounding a finite region on the plane, and we use R to denote the finite region constructed above, see Figure 4.

Let x_1 be the first vertex in \mathbb{Z}^2 along ℓ_{xy} starting from x ; and let p_1 be the first vertex in \mathbb{Z}^2 along ℓ_{pq} starting from p . Let x_2 (resp. p_2) be the midpoint of the line segment $[x, x_1]$ (resp. $[p, p_1]$). Since $x \in S$ and $x_1 \in C_1$, the interface of S passes x_2 . Similarly the interface of S passes p_2 as well.

We claim that x_2 and p_2 are in the same component of the interface of S . To see why this is true, consider the connected component of the interface of S passing x_2 , denoted by $\gamma_{x_2}^S$; $\gamma_{x_2}^S$ is either a cycle or an infinite path. Therefore $\gamma_{x_2}^S$ intersects $\partial R = \ell_{xy} \cup \ell_{pq} \cup \ell_{px} \cup \ell_{qy}$ an even number of times. But the only other possible intersection of $\gamma_{x_2}^S$ with ∂R is p_2 , therefore $p_2 \in \gamma_{x_2}^S$. Moreover, any other component of interface of S does not intersect ∂R .

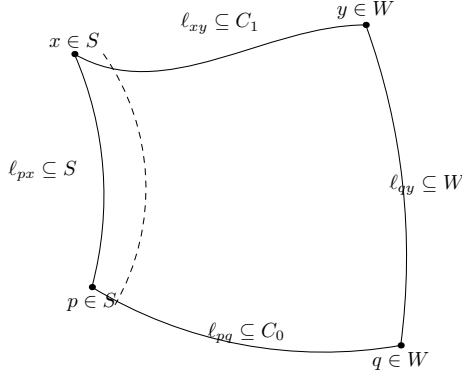


FIGURE 4. Infinite clusters and adjacent contours

Starting from x_2 , we find all the vertices in $C_1 \cap R$ satisfying one of the following conditions

- I the distance of the vertex to $\gamma_{x_2}^S \cap R$ is $\frac{1}{4}$;
- II the distance of the vertex to $\gamma_{x_2}^S \cap R$ is $\frac{\sqrt{2}}{4}$, and the vertex is incident to two vertices in $C_1 \cap R$ whose distances to $\gamma_{x_2}^S \cap R$ are $\frac{1}{4}$.

Then x_2 and p_2 , as well as all the vertices found above, are in the same cluster of the constrained site configuration on \mathbb{Z}^2 . However $x_2 \in C_1$, $p_2 \in C_0$, and C_1 and C_0 are distinct clusters. This contradiction implies Theorem 1.4.

5. NONEXISTENCE OF INFINITE CLUSTERS UNDER THE FINITE ENERGY ASSUMPTION

In this section, we prove Theorems 1.5 and 1.6.

Let μ be a probability measure on Ω' satisfying Assumptions 1.1, 1.2, 1.3. By construction ν_1 (see Sect. 1.3) is a translation invariant measure on bond configurations in Φ'_1 . Each such bond configuration on \mathbb{L} corresponds to two site configurations in $\{0, 1\}^{V(\mathbb{L}^*)}$, where $V(\mathbb{L}^*)$ is the vertex set of \mathbb{L}^* . Namely, if the state of a vertex v_0 of \mathbb{L}^* is “1”, then for any vertex v of \mathbb{L}^* , let ℓ_v be a dual path consisting of edges of \mathbb{L}^* and connecting v_0 and v . The states of each pair of neighboring vertices along ℓ_v are different if and only if the dual edge along ℓ_v crosses a present primal edge. Under such a “state change criterion”, the state at v is “1” if and only if ℓ_v crosses present primal edges an even number of times; if ℓ_v crosses present primal edges an odd number of times, the state at v is “0”. It is not hard to see that the state of v , defined above, is independent of the choice of ℓ_v , since each vertex of \mathbb{L} has an even number of incident present edges, and therefore changing the path from v_0 to v does not change the parity of intersections of the path with present primal edges. Similarly, if v_0 has state “0”, we can also determine the state of any vertex of \mathbb{L}^* according to the same “state change criterion”.

If we further assume that the vertex v_0 in \mathbb{L}^* has state “0” with probability $1/2$ and has state “1” with probability $1/2$, and the state of v_0 is independent of the bond configurations of \mathbb{L} , then ν_1 induces a translation-invariant, symmetric probability measure on site configurations of \mathbb{L}^* , denoted by μ_1 . In fact, under assumption 1.3, μ_1 has finite energy in the normal sense (see [7]). Then μ_1 -a.s. there is at most one infinite “1”-cluster and at most one infinite “0”-cluster on \mathbb{L}^* , according the arguments in [31, 7].

Recall that ν is the induced measure on contour configurations in Φ' by μ , and ν_2 is the marginal distribution of ν on bond configurations of \mathbb{L}^* , with state space Φ'_2 and σ -algebra \mathcal{F}_d . Let μ_2 be the induced translation-invariant, symmetric measure on site configurations of \mathbb{L} . By translation invariance of μ and the finite energy of μ_1 , we deduce that μ_2 also has finite energy in the normal sense.

The following cases might happen

- I there are no infinite “1”-clusters and no infinite “0”-clusters in the site configuration on vertices on \mathbb{L}^* , denote the event by E_1 ;
- II there is a unique infinite “1”-cluster, and no infinite “0”-cluster in the site configuration of \mathbb{L}^* ; or there is a unique infinite “0”-cluster, and no infinite “1”-cluster in the site configuration of \mathbb{L}^* , denote the event by E_2 ;
- III there is a unique infinite “1”-cluster, and a unique infinite “0”-cluster in the site configuration of \mathbb{L}^* ; denote the event by E_3 ;

Let $\phi \in \Phi'$ be a contour configuration. Let $\phi_p \subseteq \phi$ (resp. $\phi_d \subseteq \phi$) be the configuration of primal (resp. dual) contours, i.e., the set of all edges of \mathbb{L} (resp. \mathbb{L}^*) present in ϕ . Note that

$$\phi = \phi_p \cup \phi_d, \quad \phi_p \cap \phi_d = \emptyset.$$

The events E_1, E_2, E_3 can also be interpreted as follows: E_1 is the event that $\mathbb{R}^2 \setminus \phi_p$ has no infinite components; E_2 is the event that $\mathbb{R}^2 \setminus \phi_p$ has a unique infinite component; E_3 is the event that $\mathbb{R}^2 \setminus \phi_p$ has exactly two infinite components, and any path consisting of edges of \mathbb{L}^* connecting two vertices of \mathbb{L}^* in the two infinite components of $\mathbb{R}^2 \setminus \phi_p$ must cross the present primal contours an odd number of times.

Therefore E_1, E_2, E_3 are measurable with respect to the σ -algebra of the original site configuration process on \mathbb{Z}^2 , and are $2\mathbb{Z} \times 2\mathbb{Z}$ translation-invariant, hence by ergodicity of μ , exactly one of the following cases is true

- I $\mu(E_1) = 1$;
- II $\mu(E_2) = 1$;
- III $\mu(E_3) = 1$.

Lemma 5.1. *If $\mu(E_1) = 1$, then μ almost surely (μ -a.s.) there are no infinite “0”-clusters or infinite “1”-clusters.*

Proof. If $\mu(E_1) = 1$, then μ -a.s. $\mathbb{R}^2 \setminus \phi_p$ has no infinite components. By translation invariance, μ -a.s. $\mathbb{R}^2 \setminus \phi_d$ has no infinite components. Hence $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has no infinite components. Therefore in this case, μ almost surely (μ -a.s.) there are no infinite “0”-clusters or infinite “1”-clusters. \square

Lemma 5.2. *If $\mu(E_2) = 1$, then μ almost surely (μ -a.s.) there are no infinite “0”-clusters or infinite “1”-clusters.*

Proof. If $\mu(E_2) = 1$, then μ -a.s. $\mathbb{R}^2 \setminus \phi_p$ has exactly one infinite component. By translation invariance, μ -a.s. $\mathbb{R}^2 \setminus \phi_d$ has exactly one infinite component. We claim that $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has at most one infinite component μ -a.s..

To see why this is true, note that when $\mu(E_2) = 1$, $\mathbb{R}^2 \setminus \phi_p$ is a disjoint union of finite components and a unique infinite component. Let Q_1 be the infinite component of $\mathbb{R}^2 \setminus \phi_p$.

Since $\phi_p \cap \phi_d = \emptyset$, we can add dual contours to components of $\mathbb{R}^2 \setminus \phi_p$. Adding dual contours to finite components of $\mathbb{R}^2 \setminus \phi_p$ does not generate infinite clusters. Similarly, $\mathbb{R}^2 \setminus \phi_d$ is a disjoint union of finite components and a unique infinite component. Let Q_2 be the infinite component of $\mathbb{R}^2 \setminus \phi_d$. We can also add primal contours to components of $\mathbb{R}^2 \setminus \phi_d$, and adding primal contours to finite components of $\mathbb{R}^2 \setminus \phi_d$ does not generate infinite clusters. Therefore if $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has infinite components, each infinite component must be a component of $Q_1 \cap Q_2$.

We claim that if $Q_1 \cap Q_2$ is nonempty, then it is infinite and connected. Here by “infinite”, we mean $Q_1 \cap Q_2 \cap \mathbb{Z}^2$ has infinitely many vertices; by “connected”, we mean for any $x \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$, $y \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$, we can find a path in $Q_1 \cap Q_2$ connecting x and y , consisting of edges of G , and every vertex along the path is in $Q_1 \cap Q_2 \cap \mathbb{Z}^2$.

Let $D_1 \subseteq \phi_p$ be all the primal contours adjacent to Q_1 , and let I_1 be the interface of D_1 within $Q_1 \cap Q_2$. Let $D_2 \subseteq \phi_d$ be all the dual contours adjacent to Q_2 , and let I_2 be the interface of D_2 within $Q_1 \cap Q_2$. Note that $I_1 \cap I_2 = \emptyset$.

We first show that $Q_1 \cap Q_2$ is connected. For any $x \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$, $y \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$, we can find a path $\ell_{xy} \subseteq Q_1$, such that ℓ_{xy} consists of edges of G and connects x and y , since Q_1 is connected. If $\ell_{xy} \cap I_2 = \emptyset$, then we find a path $\ell_{xy} \subseteq Q_1 \cap Q_2$, consisting of edges of G , and connecting x and y , hence x and y are connected in $Q_1 \cap Q_2$.

If $\ell_{xy} \cap I_2 \neq \emptyset$, traveling along ℓ_{xy} from x to y , let z' be the first point in $\ell_{xy} \cap I_2$, and let w' be the last point in $\ell_{xy} \cap I_2$. Let z be the last vertex in $\mathbb{Z}^2 \cap \ell_{xy}$ before z' , when traveling along ℓ_{xy} from x to y . Let w be the first vertex in $\mathbb{Z}^2 \cap \ell_{xy}$ after w' , when traveling along ℓ_{xy} from x to y . Then $z, w \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$. Since Q_2 is connected, there exists a path γ_{zw} , connecting z and w , and consisting of edges of G , and $\gamma_{zw} \cap I_2 = \emptyset$.

Consider all the components of I_2 intersecting ℓ_{xy} ; each component is either a finite cycle, or an infinite self-avoiding path. There are finitely many components of I_2 intersecting ℓ_{xy} , denoted by S_1, \dots, S_k , with $k \geq 1$. For $1 \leq i \leq k$, if S_i is a finite cycle, then $\mathbb{R}^2 \setminus S_i$ has exactly one finite component and one infinite component. Let R_i be the finite component of $\mathbb{R}^2 \setminus S_i$, and R'_i be the infinite component. We have $x, y, z, w \in R'_i$, because $x, y, z, w \in Q_2$, and Q_2 is an infinite component of $\mathbb{R}^2 \setminus \phi_d$, and cannot be bounded in the finite region R_i . If S_i is an infinite self-avoiding path, then $\mathbb{R}^2 \setminus S_i$ has exactly two infinite components. Let R'_i be the infinite component of $\mathbb{R}^2 \setminus S_i$ including γ_{zw} , and R_i be the other infinite component.

Define

$$\tilde{R}_i = \{\xi \in r : r \text{ is a unit square of } G, r \cap R_i \neq \emptyset\}.$$

In other words, \tilde{R}_i is the closed set consisting of all the points of the plane in (closed) unit squares of G intersecting R_i . Let $\tilde{R} = \cup_{i=1}^k \tilde{R}_i$, and R be the interior of \tilde{R} , ∂R be the boundary of R , then $\gamma_{zw} \cap R = \emptyset$. However, it is possible that $\gamma_{zw} \cap \partial R \neq \emptyset$.

Note that $x, y \notin R$. ℓ_{xy} is cut by intersections with ∂R into segments. We will construct a path ℓ'_{xy} as follows: starting from x , follow ℓ_{xy} until z ; then for the next segment of ℓ_{xy} in R , follow a finite path along ∂R to connect the two endpoints of the segment. This is always possible because γ_{zw} is a finite path connecting z and w , and $\gamma_{zw} \cap R = \emptyset$. Then for the next interval of ℓ_{xy} outside R , ℓ'_{xy} follow the path of ℓ_{xy} , and so on, until we arrive at y .

We can see that ℓ'_{xy} is a path connecting x and y , consisting of edges of G , and $\ell'_{xy} \cap (I_1 \cup I_2) = \emptyset$. Hence $\ell'_{xy} \in Q_1 \cap Q_2$, therefore $Q_1 \cap Q_2$ is connected.

To show that $Q_1 \cap Q_2 \cap \mathbb{Z}^2$ is infinite, it suffices to show that for any $x \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$, x is in an infinite component of $\mathbb{R}^2 \setminus [I_1 \cup I_2]$. Conversely, if there exists $x \in Q_1 \cap Q_2 \cap \mathbb{Z}^2$, such that x is in a finite component of $\mathbb{R}^2 \setminus [I_1 \cup I_2]$, then we can find a finite component $K \subseteq I_1 \cup I_2$, such that $\mathbb{R}^2 \setminus K$ consists of exactly one finite component and one infinite component, and x is in a finite component of $\mathbb{R}^2 \setminus K$. Since K is connected, and $I_1 \cap I_2 = \emptyset$, either $K \subseteq I_1$ or $K \subseteq I_2$. Without loss of generality, assume $K \subseteq I_1$, then x is in a finite component of $\mathbb{R}^2 \setminus I_1$, and $x \in Q_1 \cap \mathbb{Z}^2$; but this is impossible since $Q_1 \cap \mathbb{Z}^2$ is infinite and connected. The contradiction implies that if $Q_1 \cap Q_2$ not empty, then it is infinite and connected, and $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has at most one infinite component.

If $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has no infinite component, then there are no infinite “0”-clusters or infinite “1”-clusters. If $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has exactly one infinite component, then the total number of infinite “0”-clusters and infinite “1”-clusters is 1, but this event has probability 0 since by symmetry, μ -a.s. there are the same number of infinite “0”-clusters and infinite “1”-clusters, and the total number of infinite “0”-clusters and infinite “1”-clusters is even. The contradiction implies that if $\mu(E_2) = 1$, then μ -a.s. there is no infinite “0”-clusters or infinite “1”-clusters. \square

Lemma 5.3. $\mu(E_3) = 0$.

Proof. If $\mu(E_3) = 1$, then μ -a.s. $\mathbb{R}^2 \setminus \phi_p$ has exactly two infinite components. Let R_1 and R_2 be the two infinite components of $\mathbb{R}^2 \setminus \phi_p$. By translation invariance, μ -a.s. $\mathbb{R}^2 \setminus \phi_d$ has exactly two infinite components. Let R_3 and R_4 be the two infinite components of $\mathbb{R}^2 \setminus \phi_d$.

Using the same argument as in the case $\mu(E_2) = 1$, we deduce that any infinite component of $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ must be a component of one of the following four sets: (a) $R_1 \cap R_3 := D_1$; (b) $R_2 \cap R_3 := D_2$; (c) $R_1 \cap R_4 := D_3$; (d) $R_2 \cap R_4 := D_4$. In particular D_1, D_2, D_3, D_4 are disjoint. Again using the same argument as in the case $\mu(E_2) = 1$, we deduce that for $1 \leq i \leq 4$, either $D_i = \emptyset$, or D_i is infinite and connected.

First of all, we claim that at least one of D_1, D_2, D_3, D_4 is empty. Assume none of them are empty, and we will find a contradiction. Let $x \in D_3 \cap \mathbb{Z}^2$, $y \in D_1 \cap \mathbb{Z}^2$, $z \in D_2 \cap \mathbb{Z}^2$, $w \in D_4 \cap \mathbb{Z}^2$. We have a path $\ell_{xy} \subseteq R_1$ connecting x and y , a path $\ell_{yz} \subseteq R_3$ connecting y and z , a path $\ell_{zw} \subseteq R_2$ connecting z and w , and a path $\ell_{xw} \subseteq R_4$ connecting x and w ; moreover, $\ell_{xy}, \ell_{yz}, \ell_{zw}, \ell_{xw}$ consist of edges of \mathbb{Z}^2 .

Consider all the dual contours intersecting ℓ_{xy} . Recall that any dual contour configuration (bond configuration on $E(\mathbb{L}^*)$ satisfying the condition that each vertex of \mathbb{L}^* has an even number of incident present edges) induces two site configurations on $V(\mathbb{L})$; namely a pair of neighboring vertices $u, v \in V(\mathbb{L})$ have different states if and only if the edge $\langle u, v \rangle \in E(\mathbb{L})$ intersects a present dual edge. This correspondence is 1-to-2 because any two site configurations ω, ω' of $V(\mathbb{L})$, such that the vertices with state “1” in ω' have state “0” in ω and vice versa, correspond to the same contour configuration. Although x and y are vertices of \mathbb{Z}^2 , instead of vertices of \mathbb{L} , by the construction of the lattice \mathbb{L} , we know that each vertex of \mathbb{L} is located at the center of a white face of \mathbb{Z}^2 , such that the lower right corner of the white face has coordinate $(2m, 2n)$, where m and n are integers. Any

site configuration on $V(\mathbb{L})$ has a natural corresponding site configuration on \mathbb{Z}^2 . Namely, any four vertices of \mathbb{Z}^2 surrounding a white face with lower right corner $(2m, 2n)$, have the same state as the vertex of \mathbb{L} at the center of the white face. Two neighboring vertices $u, v \in \mathbb{Z}^2$ have different states if and only if the edge $\langle u, v \rangle$ intersects a present edge of \mathbb{L}^* in the dual contour configuration.

Since x and y have different states in the induced site configuration on \mathbb{Z}^2 by the dual contour configurations, the total number of intersections of ℓ_{xy} with dual contours is odd. However, we will show that the total number of intersections of ℓ_{xy} with dual contours is even. This contradiction implies that at least one of D_1, D_2, D_3, D_4 is empty. More precisely, we will show that the number of intersections of ℓ_{xy} with each single dual contour is even. Since ℓ_{xy} intersects finitely many dual contours in total, this implies that the total number of intersections of ℓ_{xy} with dual contours is even.

First of all, let C_f be a finite dual contour intersecting ℓ_{xy} . Using the same argument as in the proof of Lemma 3.2, we deduce that x and y are in the unique infinite component of $\mathbb{R}^2 \setminus C_f$; and ℓ_{xy} intersects C_f an even number of times.

Now let η be an infinite dual contour intersecting ℓ_{xy} , and adjacent to both R_3 and R_4 . Since $\ell_{xy} \in R_1$, and $\eta \cap \ell_{xy} \neq \emptyset$, we have $\eta \cap R_1 \neq \emptyset$. The boundaries of the region R_1 are primal contours, and primal contours and dual contours can never intersect, we have $\eta \subseteq R_1$.

We will show that η intersects ℓ_{xy} an even number of times. Since η is a dual contour, i.e. each vertex of \mathbb{L}^* has an even number of incident present edges in η , η intersects the closed path $\ell_{xy} \cup \ell_{yz} \cup \ell_{zw} \cup \ell_{xw}$ an even number of times. But $\eta \cap \ell_{xw} = \emptyset$, because $\ell_{xw} \subseteq R_4$, and η is adjacent to R_4 ; $\eta \cap \ell_{yz} = \emptyset$, because $\ell_{yz} \subseteq R_3$, and η is adjacent to R_3 ; $\eta \cap \ell_{zw} = \emptyset$, because $\ell_{zw} \in R_2$, $\eta \subseteq R_1$, and $R_1 \cap R_2 = \emptyset$. Therefore η intersects ℓ_{xy} an even number of times.

Let η_1 be an infinite dual contour intersecting ℓ_{xy} , and adjacent to exactly one of R_3 and R_4 . We will show that η_1 intersects ℓ_{xy} an even number of times. Without loss of generality, assume η_1 is adjacent to R_4 , but not R_3 . The other case can be proved using exactly the same arguments. Moving along ℓ_{xy} from x to y , let z be the last point in $\ell_{xy} \cap \eta_1$. Let p_{zy} be the portion of ℓ_{xy} between z and y . Let z' be the first vertex in $\mathbb{Z}^2 \cap p_{zy}$ when moving along p_{zy} from z to y .

Using the same arguments as in the proof of Lemma 4.3, we deduce that if p_{zy} does not intersect infinite dual contours at interior points, then y and z' are in the same component of $\mathbb{R}^2 \setminus \phi_d$. Then η_1 is adjacent to R_3 , since $y, z' \in R_3$, and η_1 is adjacent to z' . This contradiction implies that p_{zy} must intersect infinite dual contours at interior points.

Let ξ be an infinite dual contour intersecting p_{zy} at interior points, and let w be the first point in $p_{zy} \cap \xi$, when moving along p_{zy} from z to y . Let p_{zw} be the portion of p_{zy} between z and w . Let w' be the last point in \mathbb{Z}^2 when moving along p_{zw} from z to w . Assume p_{zw} does not intersect infinite dual contours at interior points. Then z' and w' are in the same component of $\mathbb{R}^2 \setminus \phi_d$. Moreover, using the same arguments as in the proof of Lemma 3.2, we infer that z' and w' are in the same infinite component of $\mathbb{R}^2 \setminus \phi_d$. Since $\mathbb{R}^2 \setminus \phi_d$ has exactly two infinite components, R_3 and R_4 , we have either $z', w' \in R_3$, or $z', w' \in R_4$. But if $z', w' \in R_3$, then η_1 is adjacent to R_3 , this is impossible by assumption.

Therefore we have $z', w' \in R_4$; note that $x \in R_4$, hence z' and x are in the same component of $\mathbb{R}^2 \setminus \eta_1$, therefore η_1 intersects ℓ_{xy} an even number of times.

Next we will show that it is impossible to have an infinite dual contour intersecting ℓ_{xy} , and adjacent to neither R_3 nor R_4 . Assume such a contour exists, denoted by η_2 , and we will derive a contradiction.

Let z be the last vertex in $\ell_{xy} \cap \eta_2$, when moving along ℓ_{xy} from x to y , and let p_{zy} be the portion of ℓ_{xy} between z and y . If p_{zy} does not intersect infinite dual contours at interior points, then η_2 is adjacent to R_3 , this is impossible.

Let ξ be an infinite dual contour intersecting p_{zy} at interior points, and let w be the first vertex in $p_{zy} \cap \xi$, when moving along p_{zy} from z to y . Let p_{zw} be the portion of p_{zy} between z and w , and assume p_{zw} does not intersect infinite dual contours at interior points. Let z' (resp. w') be the first (resp. last) vertex in $p_{zw} \cap \mathbb{Z}^2$, when moving along p_{zw} from z to w . Then using the arguments in the proof of Lemma 3.2, we infer that z' and w' are in the same infinite components of $\mathbb{R}^2 \setminus \phi_d$. This component is distinct from R_3 and R_4 , because η_2 is incident to neither R_3 nor R_4 . Then there exist at least 3 infinite components of $\mathbb{R}^2 \setminus \phi_d$. This is a contradiction to the fact that $\mathbb{R}^2 \setminus \phi_d$ has exactly two infinite components.

Now we complete the proof that at least one of D_1, D_2, D_3, D_4 is empty.

Moreover, exactly one of D_1, D_2, D_3, D_4 is empty. To see why this is true, using the same arguments as in the proofs of Lemmas 3.2 and 4.3, we deduce that μ -a.s. there exists an infinite primal contour adjacent to both R_1 and R_2 , denoted by ξ_1 ; and μ -a.s. there exists an infinite dual contour adjacent to both R_3 and R_4 , denoted by ξ_2 . Since primal contours and dual contours do not intersect, one of the following cases is true: (a) $\xi_1 \subseteq R_3$, $\xi_2 \subseteq R_1$; (b) $\xi_1 \subseteq R_4$, $\xi_2 \subseteq R_1$; (c) $\xi_1 \subseteq R_3$, $\xi_2 \subseteq R_2$; (d) $\xi_1 \subseteq R_4$, $\xi_2 \subseteq R_2$. Without loss of generality, assume $\xi_1 \subseteq R_3$, $\xi_2 \subseteq R_1$, then $R_3 \cap R_2 \neq \emptyset$, $R_3 \cap R_1 \neq \emptyset$, and $R_1 \cap R_4 \neq \emptyset$. Therefore when $\xi_1 \subseteq R_3$, $\xi_2 \subseteq R_1$, $R_2 \cap R_4 = \emptyset$, because at least one of D_1, D_2, D_3, D_4 is empty. Similar for all the other cases.

Therefore, if $\mu(E_3) = 1$, μ -a.s. $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has exactly 3 infinite components. However the number of infinite components of $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ is always even because μ -a.s. there are the same number of infinite “0”-clusters and infinite “1”-clusters by symmetry. Hence $\mu(E_3) = 0$.

□

5.1. Proof of Theorem 1.5. Part I follows from Lemmas 5.1-5.3, and Part II follows from Part I by Corollary 3.3. Now we prove Part III.

Assume there exists an infinite $*$ -cluster, denoted by $C_{\infty,*}$. Since there are no infinite clusters, $C_{\infty,*}$ is almost surely a union of infinitely many finite clusters, i.e.

$$C_{\infty,*} = \cup_{i=1}^{\infty} C_{f_i},$$

where C_{f_i} is a finite cluster.

Let the boundary of C_{f_i} , ∂C_{f_i} , consist of edges in $E(\mathbb{L}) \cup E(\mathbb{L}^*)$ satisfying the following conditions

- I $e \in E(\mathbb{L}) \cup E(\mathbb{L}^*)$ is present in a contour;
- II there exists a vertex $v \in C_{f_i} \cap \mathbb{Z}^2$, such that $\text{dist}(v, e) = \frac{1}{2}$.

Let $\partial_o C_{f_i} \subseteq \partial C_{f_i}$ be a subset of ∂C_{f_i} , satisfying the following conditions

- I $\partial_o C_{f_i}$ is a cycle, i.e. each vertex along $\partial_o C_{f_i}$ has two incident edges in $\partial_o C_{f_i}$;
- II winding around $\partial_o C_{f_i}$ counterclockwise, C_{f_i} is on the left of $\partial_o C_{f_i}$.

It is not hard to see that there is a unique cycle $\partial_o C_{f_i} \subseteq \partial C_{f_i}$ satisfying the above conditions. We call $\partial_o C_{f_i}$ the outer boundary of C_{f_i} .

Since the infinite $*$ -cluster $C_{\infty,*}$ is the union of infinitely many finite clusters. Starting from $\partial_o C_{f_1}$, we can find $C_{f_{i_1}}$, such that $\partial_o C_{f_1} \cap \partial_o C_{f_{i_1}} \neq \emptyset$. Moreover, we can find $C_{f_{i_2}}$, such that $\partial_o C_{f_{i_2}} \cap (\partial_o C_{f_1} \cup \partial_o C_{f_{i_1}}) \neq \emptyset$, and so on. This way we obtain an infinite contour including $\cup_{j=1}^{\infty} \partial_o C_{f_{i_j}}$. But μ -a.s. infinite contours do not exist, we conclude that μ -a.s. infinite $*$ -clusters do not exist.

5.2. Proof of Theorem 1.6. Let E_1, E_2, E_3 be events given at the beginning of this section. It suffices to prove that $\mu(E_1) = 1$, which is equivalent to prove that $\mu(E_2) = 0$ and $\mu(E_3) = 0$. By Lemma 5.3, we only need to prove $\mu(E_2) = 0$. By ergodicity of E_2 , it suffices to prove that $\mu(E_2) \neq 1$.

If $\mu(E_2) = 1$, let Q_1 (resp. Q_2) be the unique infinite component of $\mathbb{R}^2 \setminus \phi_p$ (resp. $\mathbb{R}^2 \setminus \phi_d$). We will prove that $Q_1 \cap Q_2$ is nonempty here; then according to the proof of Lemma 5.2, $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has exactly one infinite component; but this is not possible since $\mathbb{R}^2 \setminus [\phi_p \cup \phi_d]$ has no infinite components according to Theorem ??.

Assume $Q_1 \cap Q_2 = \emptyset$, we will derive a contradiction. For vertices of \mathbb{Z}^2 , we color them (randomly) by red or blue according to the following rule: a vertex is colored red (resp. blue) if it is in an infinite (resp. finite) component of $\mathbb{R}^2 \setminus \phi_p$. A red (resp. blue) cluster is a connected set of vertices of \mathbb{Z}^2 , in which every vertex is red (resp. blue). It is not hard to see that, under the assumption that $\mu(E_2) = 1$, there is a unique red cluster, and the red cluster is infinite.

Let C_b be a blue cluster. We define the boundary of C_b to be the set of edges of \mathbb{L} , in which each edge separates a vertex in C_b and a neighboring vertex in the infinite red cluster. Under the assumption that $\mu(E_2) = 1$, we can see that every blue cluster is simply-connected, i.e. the boundary has only one component. That is because if we have a blue cluster which is not simply-connected, then we will have at least two red clusters, this is a contradiction to the fact that, there is a unique red cluster.

If there is an infinite blue cluster, since the infinite blue cluster is simply-connected, and there is an infinite red cluster as well, the infinite blue cluster must have infinite boundary. To see why this is true, let $v_1 \in \mathbb{Z}^2$ be a vertex in an infinite blue cluster incident to vertex $v_2 \in \mathbb{Z}^2$ in the infinite red cluster, then the midpoint z of v_1 and v_2 is on the boundary of an infinite blue cluster, and the boundary of the infinite blue cluster is a connected set of edges including z , although the boundary may not be a complete contour. If the boundary of the infinite blue cluster is finite, then either the infinite blue cluster or the infinite red cluster is finite, but this is not possible. Therefore the boundary of the infinite blue cluster is infinite.

In a similar way we (randomly) color a vertex of \mathbb{Z}^2 by yellow (resp. green), if the vertex is in an infinite (resp. finite) component of $\mathbb{R}^2 \setminus \phi_d$. Similarly we define yellow clusters and green clusters. Under the assumption that $\mu(E_2) = 1$, there is a unique yellow cluster, and

the yellow cluster is infinite. All the green clusters are simply connected. If there is an infinite green cluster, then the infinite green cluster has infinite boundary.

If $Q_1 \cap Q_2 = \emptyset$, since Q_2 and Q_1 are connected, Q_2 must be a subset of a blue cluster, and Q_1 must be a subset of a green cluster. Since Q_1 and Q_2 are infinite, we infer that there is an infinite blue cluster, as well as an infinite green cluster. According to the arguments above, the infinite blue (resp. green) cluster has an infinite connected boundary, consisting of present primal (resp. dual) edges in the contour configuration. Since present primal edges and present dual edges can never intersect, we infer that the infinite boundary of the infinite blue cluster is a subset of the infinite yellow cluster. Then we can find a vertex $v \in \mathbb{Z}^2$ in the infinite red cluster, incident to the infinite boundary of the infinite blue cluster, hence v is in the infinite yellow cluster as well. Therefore $Q_1 \cap Q_2 \neq \emptyset$, and this completes the proof.

6. NON-SYMMETRIC CASE

In this section, we prove Theorem 1.7. Let μ be a probability measure on Ω' satisfying Assumptions 1.1 and 1.3.

Lemma 6.1. *Let μ be a probability measure as described at the beginning of this section. Then μ -a.s. the number of infinite primal contours is $0, 1, \infty$.*

Proof. The proof is inspired by [31, 29]. Let k be an integer satisfying $1 \leq k < \infty$, and E_k be the event that there exist exactly k infinite primal contours. Since μ is $2\mathbb{Z} \times 2\mathbb{Z}$ ergodic, either $\mu(E_k) = 0$ or $\mu(E_k) = 1$.

Assume there exists $k \geq 2$, such that $\mu(E_k) = 1$. Let B_n^* be the $n \times n$ box of \mathbb{L}^* , centered at the origin. Let F_n be the event that B_n intersects all the k infinite primal contours, then

$$\lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} (\cup_n F_n) = 1.$$

Hence there exists $N > 0$, such that

$$\mu(F_N) > \epsilon_0 > 0.$$

Assume F_N occurs. ∂B_N^* consists of four line segments, each of which has length $2N$. We claim that any infinite primal contour intersects ∂B_N^* an even number of times. To see why this is true, we first note that any infinite primal contour can be decomposed into a union of cycles and infinite paths, such that distinct cycles and infinite paths in the decomposition share no common edges; although the decomposition may not be unique. Fix one decomposition for an infinite primal contour. There are only finitely many cycles and infinite paths intersecting ∂B_N^* . Each cycle intersects ∂B_N^* an even number of times, and so does each infinite path.

Therefore each infinite primal contour intersects ∂B_N^* an even number of times. Since there are finitely many, namely, k infinite primal contours in total, ∂B_N^* intersects infinite primal contours an even number of times. Similarly, ∂B_N^* intersects finite primal contours an even number of times.

Let B_{N-1} be the interior dual graph of B_N^* . Namely, B_{N-1} is the $(N-1) \times (N-1)$ box of \mathbb{L} centered at the origin. Each vertex of B_{N-1} is placed at the center of a face of B_N^* ;

two vertices of B_{N-1} is connected by an edge if and only if the two corresponding faces in B_N^* share an edge.

Let z_1, \dots, z_{2l} be all the intersections of primal contours with ∂B_N^* , where $l \geq 1$ is an integer. Note that at each point z_i ($1 \leq i \leq 2l$), the primal contour is perpendicular to ∂B_N^* . Moving along the primal contour from z_i by 1 into B_N^* , we arrive at a point $z'_i \in V(\mathbb{L}) \cap \partial B_{N-1}$. Note that the vertices z'_1, \dots, z'_{2l} may not be distinct. In fact if $z'_i = z'_j$, for $i \neq j$, then z'_i is at the corner of the square B_{N-1} .

Without loss of generality, assume z'_1, \dots, z'_{2t} are all the distinct points in $\{z'_i\}_{1 \leq i \leq 2l}$, corresponding to a unique point in $\{z_i\}_{1 \leq i \leq 2l}$. Here t is an integer in $[0, l]$. Let $z'_{2t+1}, \dots, z'_{2t+s}$ ($0 \leq s \leq 4$) be all the distinct points in $\{z'_i\}_{1 \leq i \leq 2l}$ located at corners of B_{N-1} , and corresponding to two points in $\{z_i\}_{1 \leq i \leq 2l}$. Let v_1, v_2, v_3, v_4 be the four corners of B_{N-1} , and let

$$\begin{aligned} W_0 &= \{v_1, v_2, v_3, v_4\} \setminus \{z'_1, z'_2, \dots, z'_{2t+s}\} \\ W_2 &= \{z'_{2t+1}, \dots, z'_{2t+s}\} \\ W_1 &= \{v_1, v_2, v_3, v_4\} \setminus (W_0 \cup W_2). \end{aligned}$$

In other words, W_0 (resp. W_1, W_2) consists of all the points in $\{v_1, v_2, v_3, v_4\}$ which have 0 (resp. 1, 2) incident present edges outside B_{N-1} in the primal contour configuration.

We change configurations in B_{N-1} following the steps below in order.

- I Make all the edges in $B_{N-1} \setminus \partial B_{N-1}$ present, and make all the edges in ∂B_{N-1} absent. After this step, all the vertices in $W_0 \cup W_2 \cup [\{z'_1, \dots, z'_{2t}\} \setminus W_1]$ have an even number of incident present edges, as well as all the vertices in $[B_{N-1} \setminus \partial B_{N-1}] \cup [B_{N-1}]^c$. The only vertices with an odd number of incident present edges are vertices in $X_1 := [\partial B_{N-1} \setminus (W_0 \cup \{z'_1, \dots, z'_{2t}\} \cup W_2)] \cup W_1$. More precisely, after this step, every vertex in X_1 has exactly one incident present edge.
- II Note that the number of vertices in X_1 is even, denoted by w'_1, \dots, w'_{2s} , where $s \geq 0$ is an integer. Assume w'_1, \dots, w'_{2s} are labeled in cyclic order. Then we make all the edges along the path in ∂B_{N-1} connecting w'_{2j-1} and w'_{2j} present, where $1 \leq j \leq s$. Let W'_2 be the set of all vertices in W_2 , satisfying the condition that after this step, the vertex still has two incident present edges. Note that all the vertices in $W_2 \setminus W'_2$ have four incident present edges after this step. It is not hard to see that we can always label w'_1, \dots, w'_{2s} in such a way that $|W'_2| \leq 2$.
- III Consider $\{w'_1, \dots, w'_{2s}\}$. If $s = 0$, we make all the edges along ∂B_{N-1} present. If $s \geq 1$, and $W'_2 \neq \emptyset$, let $q \in W'_2$, then q has exactly two incident absent edges, and both incident absent edges of q are edges on ∂B_{N-1} . We move along ∂B_{N-1} , starting from the two incident absent edges of q , until for the first time we meet present edges. This way we obtain two paths along ∂B_{N-1} , namely p_{qq_1} , and p_{qq_2} , with endpoints q, q_1 and q, q_2 , respectively, such that all the edges along p_{qq_1} and p_{qq_2} are absent, and the edge incident to q_1 or q_2 in $\partial B_{N-1} \setminus p_{qq_1}$ or $\partial B_{N-1} \setminus p_{qq_2}$, respectively, is present. Since $s \geq 1$, q_1 and q_2 exist. We make all the edges along p_{qq_1} and p_{qq_2} present. Now both q_1 and q_2 have three incident present edges. Find a shortest path in $B_{N-1} \setminus \partial B_{N-1}$ connecting q_1 and q_2 , denoted by $\ell_{q_1 q_2}$, and make all the edges along $\ell_{q_1 q_2}$ absent. Now there is at most one vertex in W_2 with

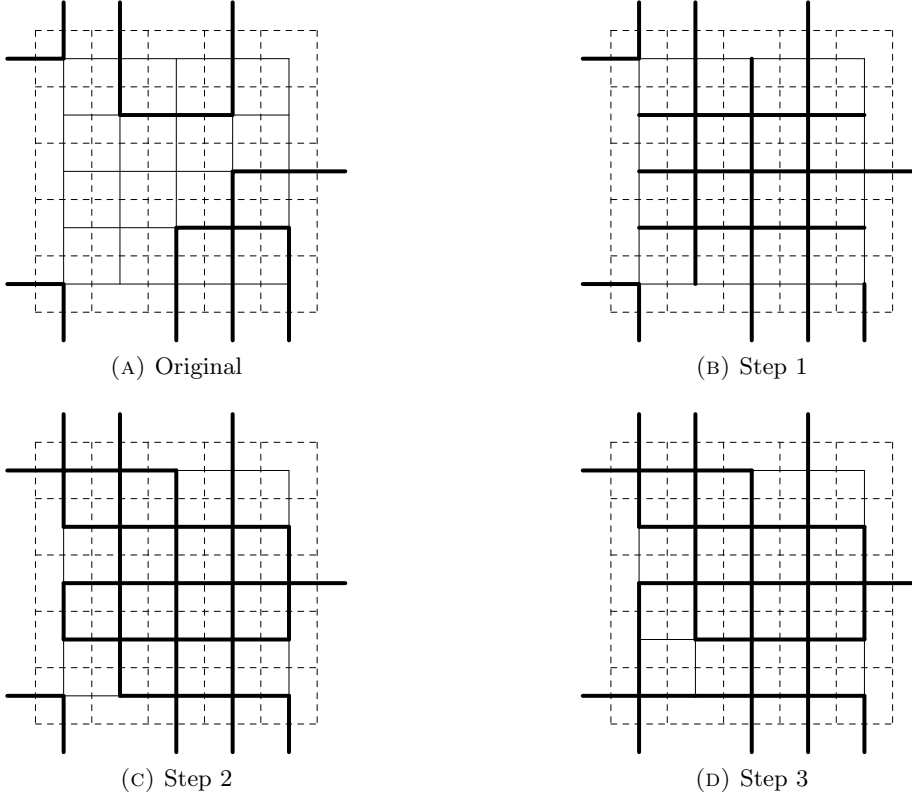


FIGURE 5. Merging finitely many infinite primal contours into one infinite primal contour

exactly two incident present edges. If such a vertex does exist, denoted by u , then we repeat the process above with respect to u , as we did for q , such that, the path $\ell_{u_1 u_2}$, consisting of edges of $B_{N-1} \setminus \partial B_{N-1}$, and along which we make every present edge absent, share no vertices with $\ell_{q_1 q_2}$. Here u_1, u_2 are vertices of \mathbb{L} on ∂B_{N-1} , obtained from u in the same way as q_1, q_2 are obtained from q .

This way after changing configurations in a finite box B_{N-1} , the new configuration has exactly one infinite primal contour, see Figure 5 for an illustration of the configuration-changing process described above to merge finitely many infinite primal contours into one infinite primal contour. According to the finite energy assumption 1.3, with a strictly positive probability, E_1 occurs. But this is a contradiction to the fact that $\mu(E_k) = 1$, for some $2 \leq k < \infty$, and the proof is complete. \square

6.1. Proof of Theorem 1.7. An $n \times n$ box B_n^* of \mathbb{L}^* is an encounter box with respect to a primal contour configuration ω , if the following two conditions hold.

- I Let B_{n-1} be the interior dual graph of B_n^* , where B_{n-1} is an $(n-1) \times (n-1)$ box of \mathbb{L} , then $B_{n-1} \cap \omega$ is connected.
- II The set $\omega \setminus B_n^*$ has no finite components, but exactly three infinite components.

Assume B_n^* is the interior dual graph of B_{n+1} , and B_{n+1} is the interior dual graph of B_{n+2}^* . Let E_∞ be the event that there exist infinitely many infinite primal contours. We claim that if $\mu(E_\infty) = 1$, then there exists $N > 0$, such that the probability that the box B_N^* centered at $(0, 0)$ is an encounter box is strictly positive.

First of all, there exists $N \geq 0$, such that the probability that B_N^* intersects at least three infinite primal contours is strictly positive. Assume the event that B_N^* intersects at least three infinite primal contours occurs. Then the number of infinite primal contours intersecting B_{N+2}^* is at least three. Let C_1, \dots, C_r be all the infinite components of $\omega \setminus B_N^*$ intersecting ∂B_{N+2}^* , where $r \geq 3$. Assume C_1, \dots, C_r are in cyclic order, i.e., when moving around ∂B_{N+2}^* clockwise starting from some point on ∂B_{N+2}^* , we will first meet interactions with C_1 , then C_2, C_3, \dots , up to C_r .

When moving along ∂B_{N+2}^* in a way as described above, let w_1 (resp. w_3, w_5) be the first intersection of ∂B_{N+2}^* with C_1 (resp. C_2, C_3), and let w_2 (resp. w_4, w_6) be the last intersection of ∂B_{N+2}^* with C_1 (resp. C_2, C_r). Clearly, ∂B_{N+2}^* intersects only finite components of $\omega \setminus B_N^*$ in (w_2, w_3) , (w_4, w_5) , (w_6, w_1) , where (w_2, w_3) is the (open) portion of ∂B_{N+2}^* between w_2 and w_3 , in which the endpoints w_2, w_3 are not included, and similarly for (w_4, w_5) , (w_6, w_1) . Let u_2 (resp. u_4, u_6) be the intersection of ∂B_{N+2}^* in (w_2, w_3) (resp. (w_4, w_5) , (w_6, w_1)) with finite components of $\omega \setminus B_N$ nearest to w_2 (resp. w_4, w_6). If (w_2, w_3) (resp. (w_4, w_5) , (w_6, w_1)) does not intersect finite contours at all, let $u_2 = w_3$ (resp. $u_4 = w_5, u_6 = w_1$).

Recall that B_{N+1} is the interior dual graph of B_{N+2}^* . Moving along the edge of \mathbb{L} centered at w_j (resp. u_i) by 1 towards B_{N+1} , we arrive at a vertex w'_j (resp. u'_i) of \mathbb{L} along ∂B_{N+1} , where $1 \leq j \leq 6$, and $i = 2, 4, 6$. There is a path along ∂B_{N+1} connecting u'_2 and w'_4 (resp. u'_4 and w'_6 , u'_6 and w'_2), passing w'_3 (resp. w'_5, w'_1), denoted by $\ell_{u'_2 w'_4}$ (resp. $\ell_{u'_4 w'_6}$, $\ell_{u'_6 w'_2}$). Make all the edges along $\ell_{u'_2 w'_4}$, $\ell_{u'_4 w'_6}$ and $\ell_{u'_6 w'_2}$ present, and preserve the configuration on all the other edges of ∂B_{N+1} , i.e., all the edges in $\partial B_{N+1} \setminus [\ell_{u'_2 w'_4} \cup \ell_{u'_4 w'_6} \cup \ell_{u'_6 w'_2}]$ are absent.

We consider each one of $\ell_{u'_2 w'_4}$, $\ell_{u'_4 w'_6}$ and $\ell_{u'_6 w'_2}$ as a closed path, i.e., the endpoints are included in the path. We preserve the configuration for all the edges connecting $\partial B_{N+1} \setminus \ell_{u'_i w'_j}$ and ∂B_{N-1} , where $(i, j) \in \{(2, 4), (4, 6), (6, 2)\}$. First make all the edges connecting $\ell_{u'_i w'_j}$ and ∂B_{N-1} absent. Then for any vertex v of \mathbb{L} along $\ell_{u'_i w'_j}$, if v has an odd number of incident present edges, make the edge incident to v , connecting $\ell_{u'_i w'_j}$, and ∂B_{N-1} present. Next we change the configuration in the box B_{N-1} in the same way as in the proof of Lemma 6.1, and obtain a new primal contour configuration ω_1 , such that $\omega \cap B_{N-1}$ is connected. It is not hard to check that B_N^* is an encounter box for ω_1 . Since from ω to ω_1 , we change configurations only in a finite box B_{N+1} , by the finite energy assumption 1.3, we conclude that the event that B_N^* is an encounter box occurs with strictly positive probability.

Using the same arguments as in [7, 29], we infer a contradiction in the number of encounter boxes in an $gN \times gN$ box, where g is an integer. Hence $\mu(E_\infty) = 0$. The Lemma then follows from Lemma 6.1.

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